Econ 204 Section 4
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Key Words

Metric Space, Normed Vector Space, Euclidean Space, Lipschitz-Equivalent, Convergence, Cluster Point, Increasing(Decreasing) Sequence, Lim Sups(Lim Infs), Rising Sun Lemma, Bolzano-Weierstrass Theorem

Section 4.1 Metric Space

- Lecture 3 Definition 1 A **metric space** is a pair (X, d), where X is a set and $d : X \times X \to$ $\mathbf{R}+$, satisfying
  1. $\forall x, y \in X \ d(x, y) \geq 0, d(x, y) = 0 \Leftrightarrow x = y$
  2. $\forall x, y \in X \ d(x, y) = d(y, x)$
  3. (triangle inequality) $\forall x, y, z \in X \ d(x, y) + d(y, z) \geq d(x, z)$

  Example 4.1.1 Let $d(x, y) = max\{|x - y|, 1\}$. Prove or disprove that $(\mathbf{R}, d)$ is a metric space.
  Disproof:
  Let $x \in X$. Then $d(x, x) = max\{|x - x|, 1\} = max\{0, 1\} = 1$. So d is not a metric.

  Example 4.1.2 Let $d(x, y) = min\{|x - y|, 1\}$. Prove or disprove that $(\mathbf{R}, d)$ is a metric space.
  Proof:: In fact this is called the standard bounded metric corresponding to d.
  Check the first two conditions for a metric. Do it by yourself.
  Check the triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$
  Now if either $|x - y| \geq 1$ or $|y - z| \geq 1$ then the right side of this inequality is at least 1; since the left side is (by definition) at most 1, the inequality holds. It remains to consider the case in which $|x - y| < 1$ and $|y - z| < 1$. In this case, we have $|x - z| \leq |x - y| + |y - z| =$ $d(x, y) + d(y, z)$. Hence $d(x, z) = min\{|x - z|, 1\} \leq |x - z| \leq d(x, y) + d(y, z)$. The triangle inequality holds.

  Example 4.1.3 Let $X = [1, +\infty)$. Let $d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$. Prove or disprove that (X, d) is a metric space.
  Proof:
  Check the first two conditions for a metric
  $\forall x, y \in X,\ d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right| \geq 0$ and $d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right| = 0 \Leftrightarrow x = y$
  $\forall x, y \in X,\ d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{1}{y} - \frac{1}{x}\right| = d(y, x)$
  Check the triangle inequality. We show that $d(x, z) \leq d(x, y) + d(y, z)$ will depend upon the ordering of x, y, and z.
  Because $d(x, z) = d(z, x)$, without loss of generality, we can assume $x \leq z$.
  Case 1. Suppose $\frac{1}{x} \geq \frac{1}{y} \geq \frac{1}{z}$. Then
  $d(x, y) + d(y, z) = \left|\frac{1}{x} - \frac{1}{y}\right| + \left|\frac{1}{y} - \frac{1}{z}\right| = \frac{1}{x} - \frac{1}{y} + \frac{1}{y} - \frac{1}{z} = \frac{1}{x} - \frac{1}{z} = \left|\frac{1}{x} - \frac{1}{z}\right| = d(x, z)$
  Case 2. Suppose $\frac{1}{x} \geq \frac{1}{z} \geq \frac{1}{y}$. Then
  $d(x, y) + d(y, z) = \left|\frac{1}{x} - \frac{1}{y}\right| + \left|\frac{1}{y} - \frac{1}{z}\right| = \frac{1}{x} - \frac{1}{y} + \frac{1}{z} - \frac{1}{y} = \frac{1}{x} + \frac{1}{z} - \frac{2}{y} \geq \frac{1}{x} + \frac{1}{z} - \frac{2}{z} = \frac{1}{x} - \frac{1}{z} =$ $\left|\frac{1}{x} - \frac{1}{z}\right| = d(x, z)$
  Case 3. Suppose $\frac{1}{y} \geq \frac{1}{x} \geq \frac{1}{z}$. Then
  $d(x, y) + d(y, z) = \left|\frac{1}{x} - \frac{1}{y}\right| + \left|\frac{1}{y} - \frac{1}{z}\right| = \frac{1}{y} - \frac{1}{x} + \frac{1}{y} - \frac{1}{x} = \frac{2}{y} - \frac{1}{x} - \frac{1}{z} \geq \frac{2}{x} - \frac{1}{x} - \frac{1}{z} = \frac{1}{x} - \frac{1}{z} =$ $\left|\frac{1}{x} - \frac{1}{z}\right| = d(x, z)$
  So the triangle inequality holds.
  Typically, showing the triangle inequality involves more effort. But do not forget to check the first two conditions.

Section 4.2 Normed Vector Space

- Lecture 2 Definition 2 Let V be a vector space over $\mathbf{R}$. A **norm** on V is a function $\| \|$ $: V \to \mathbf{R}+$ satisfying
  1. $\forall x \in V \ \|x\| \geq 0$
  2. $\forall x \in V \ \|x\| = 0 \Leftrightarrow x = 0$
  3. (triangle inequality) $\forall x, y \in V \ \|x + y\| \leq \|x\| + \|y\|$
  4. $\forall \alpha \in R, x \in V \ \|\alpha x\| = |\alpha|\|x\|$

  Example 4.2.1 $C([0, 1])$ is the set of continuous functions from $[0, 1]$ to $\mathbf{R}$. Show that $C([0, 1])$ is a normed space with norm $\|f\| = \max_{x\in[0,1]}|f(x)|$
  Solution:
  Check the first two conditions by yourself
  Check triangle inequality
  $\|f+g\| = \max_{x\in[0,1]}|f(x)+g(x)| \leq \max_{x\in[0,1]}\Big||f(x)|+|g(x)|\Big| \leq \max_{x\in[0,1]}|f(x)|+\max_{x\in[0,1]}|g(x)| = \|f\|+\|g\|$
  Check scalar multiplication
  $||\alpha f|| = \max_{x\in[0,1]}|\alpha \cdot f(x)| = \max_{x\in[0,1]}\Big||\alpha| \cdot f(x)\Big| = |\alpha| \cdot \max_{x\in[0,1]}|f(x)| = |\alpha| \cdot ||f||$

Section 4.3 Lipschitz-equivalent

- Lecture 3 Definition 5 Two norms $\| \|$ and $\| \|'$ on the same vector space V are said to be **Lipschitz-equivalent** if $\exists m, M > 0 \ \forall x \in V \ m\|x\| \leq \|x\|' \leq M\|x\|$.
- Lecture 3 Theorem 6: All norms on $\mathbf{R}^n$ are Lipschitz-equivalent.
  In exercise 6 of problem set 2, you are asked to reexamine the proof of De La Fuente.

Section 4.4 Convergence and Cluster Point

- Lecture 3 Definition 8: Let (X, d) be a metric space. A sequence x_n **converges** to x if $\forall \varepsilon > 0 \, \exists N(\varepsilon) \in \mathbf{N}$ for all $N > N(\varepsilon) \Rightarrow d(x_n, x) < \varepsilon$. This is exactly the same as the definition of convergence of a sequence of real numbers, except we replace $| \cdot |$ in $\mathbf{R}$ by the metric d.
- Lecture 3 Definition **Cluster Point**: c is a **cluster point** of a sequence $\{x_n\}$ in a metric space (X, d) if $\forall \varepsilon > 0$: $\{n : x_n \in B_\varepsilon(c)\}$ is an infinite set. Equivalently, $\forall \varepsilon > 0$, $\forall N \in \mathbf{N}$,$\exists$ $n > N$ such that $x_n \in B_\varepsilon(c)$.
- Lecture 3 Theorem 10: Let (X, d) be a metric space. $c \in X$ and $\{x_n\}$ is a sequence in X. Then c is a cluster point of $\{x_n\}$ if and only if there is a subsequence $\{x_{n_k}\}$ such that $lim_{k\to\infty} x_{n_k} = c$.

  Example 4.4.1 Uniqueness of Cluster Point.
  Prove that a convergent sequence in a metric space (X, d) has exactly one cluster point.
  Solution:
  Clearly, the limit of a convergent sequence is a cluster point of the sequence, so a convergent sequence must have at least one cluster point.
  Let x_n be a convergent sequence in a metric space (X, d), converging to x. Let P be any point different from x, so $d(x, P) > 0$. We will show that P is not a cluster point. Let $\varepsilon = \frac{d(x,P)}{2}$, so $\varepsilon > 0$. There exists $N \in \mathbf{N}$ such that for all $n > N$, $d(x_n, x) < \varepsilon$, $d(x_n, P) \geq d(x, P) - d(x_n, x) \geq 2\varepsilon - \varepsilon = \varepsilon$, so P is not a cluster point.

Section 4.5 Sequences

- Lecture 3 Definition 11: A sequence of real number x_n is **increasing (decreasing)** if $x_{n+1} \geq x_n (x_{n+1} \leq x_n)$for all n.
- Lecture 3 Theorem 13: Let $\{x_n\}$ be an increasing (decreasing) sequence of real numbers. The limit of $\{x_n\}$ exists.
- Lecture 3 Theorem 15 Lim Sups and Lim Infs Handout: Let x_n be a sequence of real numbers. Then $\lim_{n\to\infty} x_n = \gamma \in \mathbf{R} \cup \{-\infty, \infty\} \Leftrightarrow \limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \gamma$.
- Lecture 3 Theorem 16 Rising Sun Lemma: Every sequence of real numbers contains an increasing subsequence or a decreasing subsequence or both.
- Lecture 3 Theorem 17 Bolzano-Weierstrass Theorem: Every bounded sequence of real numbers contains a convergent subsequence.

Example 4.5.1 Lecture 3 Theorem 15 Lim Sups and Lim Infs Handout.

Prove this theorem for the case that γ is finite.

Solution:

($\Rightarrow$) $x_n \to \gamma \in \mathbf{R}$ implies that $\forall \varepsilon > 0$ there exist $N(\varepsilon)$ such that $n \geq N(\varepsilon) \Rightarrow |x_n - \gamma| < \varepsilon$. This means that $\gamma + \varepsilon$ is an upper bound and $\gamma - \varepsilon$ is a lower bound for $\{x_k : k \geq N(\varepsilon)\}$. Using $\alpha_n = \sup\{x_k : k \geq n\}$ and $\beta_n = \inf\{x_k : k \geq n\}$, we know that $\beta_n \leq \alpha_n$ (because a lower bound can't be greater than an upper bound) and for $n > N(\varepsilon)$,

$$\gamma - \varepsilon \leq \beta_n \leq \alpha_n \leq \gamma + \varepsilon.$$

Since this is true for any ε, it must be true that α_n and β_n both converge to x. This completes the proof that $\limsup x_n = \liminf x_n = \gamma$.

($\Leftarrow$) We will prove the contraposition. Suppose that $\lim_{n\to\infty} x_n \neq \gamma$. Then there exists an $\varepsilon > 0$ such that for all N, there is some $n \geq N$ such that $|x_n - \gamma| \geq \varepsilon$. This means that there are infinitely many x_n outside of $B_\varepsilon(\gamma)$ and it must be the case that there are infinitely many of these above $\gamma + \varepsilon$, infinitely many below $\gamma - \varepsilon$ or both. If the former is true, then $\alpha_n \geq \gamma + \varepsilon$ for all n which means that $\limsup x_n$ must be greater than or equal to $\gamma + \varepsilon$. If the latter is true, then $\beta_n \leq \gamma - \varepsilon$ for all n, so $\liminf x_n$ must be less than or equal to $\gamma - \varepsilon$. In either case, it is not true that $\limsup x_n = \liminf x_n = \gamma$, completing the proof.

Example 4.5.2 Let $x_1 = \sqrt{2}$, $x_{n+1} = \sqrt{2 + x_n}$. Prove that the sequence $\{x_n\}$ converges to 2.

Solution:

We show that the sequence is increasing and bounded, hence convergent. Then we calculate the limit.

Show that $\{x_n\}$ is strictly increasing by induction.

$x_2 = \sqrt{2 + \sqrt{2}} > \sqrt{2} = x_1$

Suppose $x_k - x_{k-1} > 0$ holds. $x_k^2 = 2 + x_{k-1}$

$x_{k+1} = \sqrt{2 + x_k} \Rightarrow x_{k+1}^2 = 2+x_k$. So $(x_{k+1} + x_k)\cdot(x_{k+1} - x_k) = x_{k+1}^2 - x_k^2 = x_k - x_{k-1} > 0$.

Since $x_n > 0$, $x_{k+1} - x_k > 0$

So $\{x_n\}$is strictly increasing.

Show that $\{x_n\}$is bounded between 0 and 3 by induction.

$0 < x_1 < 3$

Suppose $0 < x_k < 3$ holds

$0 < x_{k+1}^2 = 2 + x_k < 2 + 3 < 3^2 \Rightarrow x_{k+1} < 3$

So $\{x_n\}$ is bounded.

Hence $\{x_n\}$ converges to a finite real number x. $x_{n+1} = \sqrt{2 + x_n} \Rightarrow x = \sqrt{2 + x} \Rightarrow x = 2$

Example 4.5.3 Prove that every bounded sequence in $\mathbf{R}^2$ has a convergent subsequence.

Solution:

Let (x_n, y_n) be a bounded sequence in $\mathbf{R}^2$. Then, the coordinate sequences x_n and y_n must also be bounded sequences. By the Bolzano-Weierstrass theorem, there is a subsequence $x_{n_k} \to \alpha$. Consider now the corresponding subsequence y_{n_k}. By Bolzano-Weierstrass again, there is a further subsequence $y_{n_{k_j}} \to \beta$. Since $x_{n_{k_j}}$ is a subsequence of x_{n_j}, it converges to α, too. It follows that the subsequence $(x_{n_{k_j}}, y_{n_{k_j}}) \to (\alpha, \beta)$.

Example 4.5.4 Prove $-\sup a_n = \inf(-a_n)$

Solution:

Note that $a \le b \Rightarrow -a \ge -b$. Let $a = \inf\{-a_n\}$. Then, by definition, $a \le -a_m \ \forall m \ge n \Rightarrow -a \ge a_m, \ \forall m \ge n$. This implies that $\sup a_n \le -\inf\{-a_n\} = -a$. To show the reverse inequality, pick any $\varepsilon > 0$. Then, by the definition of the infimum, $\exists N > n$ such that $-a_N < a + \varepsilon \Rightarrow a_N > -a - \varepsilon \Rightarrow \sup a_n > -a - \varepsilon$. As $\varepsilon > 0$ may be chosen to be arbitrarily small, we obtain $\sup a_n \ge -\inf\{-a_n\} \Rightarrow \sup a_n = -\inf\{-a_n\}$. The proof that $-\inf a_n = \sup\{-a_n\}$ follows along similar lines and is omitted.