Section 8
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Key Words

Separated, Connected, Correspondence, Upper Hemicontinuous Continuous, Lower Hemicontinuous Continuous, Closed-Valued, Compact-Valued, Closed Graph

Section 8.1 Separated and Connected

- **Lecture 4 Definition 4:** $\overline{A}$: the closure of A, the smallest closed set containing A (the intersection of all closed sets containing A)
- **Lecture 7 Definition 1:** Two sets A, B in a metric space are **separated** if $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.
- **Lecture 7 Definition 1:** A set in a metric space is **connected** if it cannot be written as the union of two nonempty separated sets.
- **Lecture 7 Theorem 2:** A set S of real numbers is connected if and only if it is an interval.
- **Lecture 7 Theorem 3:** Let X be a metric space, $f : X \to Y$ continuous. If $C \subseteq X$ is connected, then $f(C)$ is connected.

Example 8.1.1 Let $\{S_i\}, i \in I$, be a collection of connected subsets of a space X. Suppose there exists an $i_0 \in I$ such that for each $i \in I$, the sets S_i and S_{i_0} have non-empty intersection. Show that $\cup_{i \in I} S_i$ is connected.
Solution:
Assume it is not true. Let U, V be nonempty separated sets in X with $U \cup V = \cup_{i\in I} S_i$, $\overline{U} \cap V = U \cap \overline{V} = \emptyset$. We can show that for every i, $U \cap S_i = S_i$ or $U \cap S_i = \emptyset$. To see this, note that $U \cap \overline{V} = \emptyset \Rightarrow (U \cap S_i) \cap \overline{(V \cap S_i)} \subseteq U \cap \overline{V} = \emptyset$. Similarly, $\overline{(U \cap S_i)} \cap (V \cap S_i) = \emptyset$. Since S_i is connected for every i, we have $S_i \cap U = \emptyset$ or $S_i \cap U = S_i$. Similarly, for every i, we also have $V \cap S_i = S_i$ or $V \cap S_i = \emptyset$. Furthermore, since $U, V \neq \emptyset$, $\exists m, n$ such that $U \cap S_m = S_n$ and $V \cap S_n = S_n$. Since $S_m \cap S_{i_0} \neq \emptyset \Rightarrow U \cap S_{i_0} \neq \emptyset \Rightarrow U \cap S_{i_0} = S_{i_0}$ and similarly we have $V \cap S_{i_0} = S_{i_0}$. Hence we have $U \cap V \neq \emptyset$. Contradiction.

Example 8.1.2 Prove that if a metric space X contains a non-empty subset A ($A \neq X$) that is both open and closed, then X is disconnected.
Solution:
We claim that A and A^c are separated. Well, A closed implies that $\overline{A} = A$, and A open implies that A^c is closed so that $\overline{A^c} = A^c$ as well. Then $\overline{A} \bigcap A^c = A \bigcap A^c = A \bigcap \overline{A^c}$. Since $X = A \cup A^c$, we see that we've written X as the union of two non-empty separated sets, and so X is disconnected.

Example 8.1.3 Suppose that the sets C and D are two non-empty separated subsets of X whose union is X. Y is a connected subset of X. Prove that Y lies entirely within C or D.
Solution:
C and D are separated $\Rightarrow \overline{C} \cap D = \emptyset$. Hence $(\overline{C} \cap Y) \cap (D \cap Y) = \emptyset$ and $(\overline{C} \cap Y) \cup (D \cap Y) = Y$. The same goes for $C \cap Y$ and $\overline{D} \cap Y$. For each of these pairs, both sets being non-empty contradicts the fact that Y is connected. Therefore one of them is empty. Therefore Y lies entirely within C or D.

Example 8.1.4 Show that if S is a connected subset of X, then so is its closure $\bar{S}$.
Solution:
By contradiction. Suppose $\bar{S}$ is not connected, then $\exists$ two nonempty disjoint sets A and B $s.t. \bar{A} \cap B = A \cap \bar{B} = \phi$ and $\bar{S} = A \cup B$. But since S is connected and $S \subseteq \bar{S} \implies S$ should be in either A or B, as a result of the Example 8.1.3. Without loss of generality, suppose $S \subseteq A$. $\implies \bar{S} \subseteq \bar{A}$. However, since $\bar{A} \cap B = \phi \implies \bar{S} \cap B = \phi$. But we know $\bar{S} = A \cup B$ and B is nonempty, so $\bar{S} \cap B \neq \phi$. Contradiction.

Section 8.2 Correspondence

- Lecture 7 Definition 5: A correspondence $\Psi : X \to Y$ is a function from X to 2^Y.
- Lecture 7 Definition 9: Let $X \subseteq E_n$, $Y \subseteq E_m$. Suppose $\Psi : X \to Y$ is a correspondence.

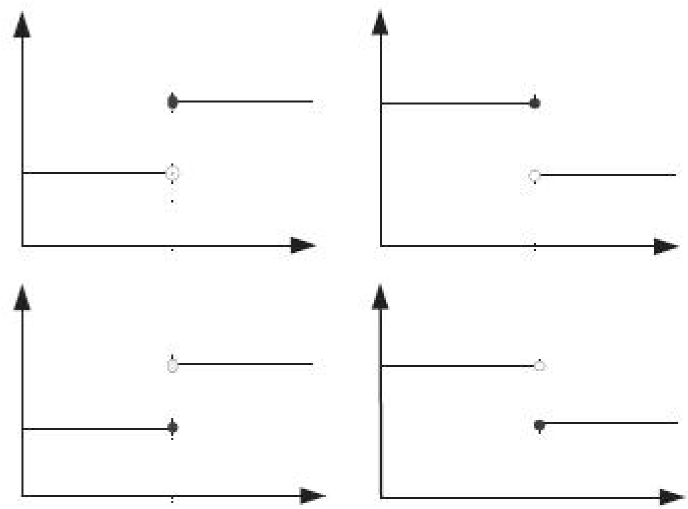
  Ψ is **upper hemicontinuous** (uhc) at $x_0 \in X$ if, for every open set $V \supseteq \Psi(x_0)$, there is an open set U with $x_0 \in U$ such that $\Psi(x) \subseteq V$ for every $x \in U \cap X$.

  Ψ is **lower hemicontinuous** (lhc) at $x_0 \in X$ if, for every open set V such that $\Psi(x_0) \cap V \neq \emptyset$, there is an open set U with $x_0 \in U$ such that $\Psi(x) \cap V \neq \emptyset$ for every $x \in U \cap X$.

  Ψ is **continuous** at $x_0 \in X$ if it is both uhc and lhc at x_0.

  Ψ is **closed** (has **closed graph**) if its graph $\{(x, y) : y \in \Psi(x)\}$ is a closed subset of $X \times E^m$.

Example 8.2.1 Determine whether the following functions "jump upward" or "jump downward'?



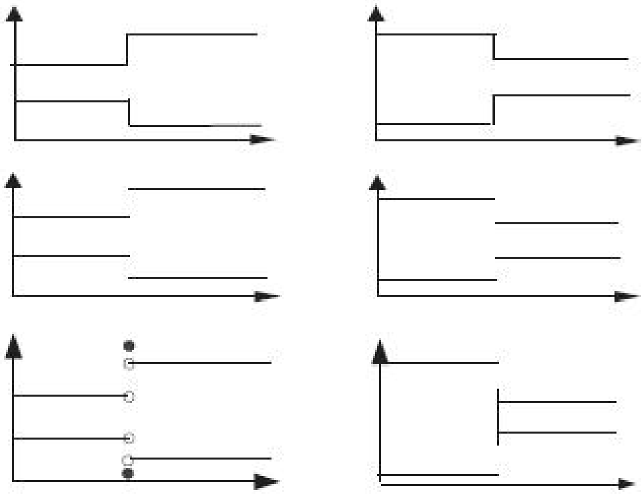
The top two are functions that "jump upward"

The bottom two are functions that "jump downward".

A function "jumps up" at the point x_0 means that the function value suddenly gets bigger when the x approaches x_0 from some direction– "explodes" the limit.

In comparison, a function "jumps down" at the point x_0 means that the function value suddenly gets smaller when the x approaches x_0 from some direction – "implodes" in the limit.

Example 8.2.2 Determine whether the following correspondences are uhc, lhc, or neither?



The top two are uhc;
The middle two are lhc.
The bottom left is uhc
The bottom right is neither uhc nor lhc.

Section 8.3 Upper Hemicontinuous

- Definition 11 Suppose $X \subseteq E_m$, $Yh \subseteq E_n$. A correspondence $\Psi : X \to Y$ is called closed-valued if $\Psi(x)$ is a closed subset of En for all x; Ψ is called compact-valued if $\Psi(x)$ is compact for all x.
- Theorem 10 Let $X \subseteq E_m$, $Y \subseteq E_n$, $f : X \to Y$ a function. Let $\Psi(x) = f(x)$ for all $x \in X$. Then $\Psi(x)$ is uhc if and only if f is continuous.
- Theorem 12 (Not in de la Fuente) Let $X \subseteq E_m$, $Y \subseteq E_n$, and $\Psi : X \to Y$ is a correspndence. • If Ψ is closed-valued and uhc, then Ψ has closed graph. If Y is compact and Ψ has closed graph, then Ψ is uhc.
- Theorem 13 (11.2) Suppose $X \subseteq E_m$, $Y \subseteq E_n$. A compact-valued correspondence $\Psi : X \to Y$ is uhc at $x_0 \in X$ if and only if, for every sequence $x_n \to x_0$, $\{x_n \subseteq X\}$, and every sequence $\{y_n\}$ such that $y_n \in \Psi(x_n)$, there is a convergent subsequence $\{y_{n_k}\}$ such that $\lim y_{n_k} \Psi(x_0)$.

Example 8.3.1 Let $X \subseteq \mathbf{E}^n, Y \subseteq \mathbf{E}^m$. A correspondence $\Psi : X \to Y$. Show that $W = \{x \in X : V \supseteq \Psi(x)\}$ is open in X for each open set V in Y if and only if Ψ is uhc.
Solution:
$\Rightarrow$: We have to find a valid U for every open set $V \supseteq \Psi(x_0)$. In this question, just take W as U. Then for every open set $V \supseteq \Psi(x_0)$, we know that there exists a set W which satisfies three properties: (1).W is open. (2) $x_0 \in W$ (3) $\forall\, x \in U \cap X \Rightarrow V \supseteq \Psi(x)$. Hence Ψ is uhc.
$\Longleftarrow$: $\forall w \in W \Rightarrow \Psi(w) \subseteq V$. Since Ψ is uhc, there is an open set U with $w \in U$ such that

$$V \supseteq \Psi(x) \text{ for every } x \in U \cap X$$

And since U is open, we can find a small ball $B_\varepsilon(w)$ such that $B_\varepsilon(w) \subseteq U$, and for every $x \in B_\varepsilon(w)$ we have $V \supseteq \Psi(x) \Rightarrow x \in W$ by the definition of W. Therefore, $B_\varepsilon(w) \subseteq W$ thus W is open.

Example 8.3.2 Consider an economy with two goods, x and y. Fix the price of good y equal to 1, and fix the consumer's income equal to I. Let p be the price of good x. Show that the budget set of consumer $B(p) = \{x \geq 0, y \geq 0, px + y \leq I\}$ is an upper hemicontinuous correspondence for $p \in (0, \infty)$.

Solution:

Use theorem 12 in Lecture 7 to finish the proof. To use theorem 12 we have to show that the correspondence has closed graph and Y is compact. First we show that the correspondence has closed graph: Let $p_n \to p \in (0, \infty)$, $(x_n, y_n) \in B(p_n)$, $(x_n, y_n) \to (x, y)$. We need to show that $(x, y) \in B(p)$. Note: $(x_n, y_n) \to (x, y)$ means $x_n \to x$ and $y_n \to y$. We know that for every n, $p_n x_n + y_n \leq I$, so $px + y = \lim(p_n x_n + y_n) \leq I$. We also have $x = \lim x_n \geq 0$ and $y = \lim y_n \geq 0$, so $(x, y) \in B(p)$, therefore B has closed graph.

Now we show that Y is compact. Since uhc is a pointwise property, we consider an arbitrary point $p \in (0, \infty)$, and find $a, b \in (0, \infty)$ such that $p \in (a, b)$. Then for all $q \in [a, b]$, $B(q) \subseteq \{x \geq 0, x \leq \frac{I}{a}, y \geq 0, y \leq I\}$, which is closed and bounded in $\mathbf{R}^2$ hence compact. So, according to Theorem 12 in Lecture 7, B is uhc at p