Determinants of Levered Portfolio Performance

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April 21, 2014[§]

Forthcoming in *Financial Analysts Journal*

Abstract

The cumulative return to a levered strategy is determined by five elements that fit together in a simple, useful formula. A previously undocumented element is the covariance between leverage and excess return to the fully invested *source* portfolio underlying the strategy. In an empirical study of volatility-targeting strategies over the 84-year period 1929–2013, this covariance accounted for a reduction in return that substantially diminished the Sharpe ratio in all cases.

Key terms: Leverage; Sharpe ratio; source portfolio; trading cost; financing cost; unintended market timing; magnified source return; excess borrowing return; risk parity; pension fund; hedge fund; fixed leverage; dynamic leverage; volatility target

1   A Simple Two-Period Example

In this paper, we show that there are five elements that determine the cumulative return to a levered strategy, and they fit together in a simple, useful formula. Looking backward, our formula can be used to attribute performance. Looking forward, an investor can populate our formula with his or her forecast of the five determinants in order to generate a forecast for return to the levered strategy.

A levered strategy begins with a fully-invested source portfolio, such as unlevered risk parity, unlevered minimum variance, or unlevered bonds. The source portfolio is then levered according to a leverage rule. The most common leverage rules target volatility: they estimate the current volatility of the source portfolio in some way, and then choose leverage so that the estimated volatility of the levered strategy matches the target. Since the *source* portfolio typically exhibits variable volatility, volatility targeting requires *dynamic* leverage, even if the volatility target is *fixed*.

Much of our intuition about levered strategies comes from single-period models. In a single-period model, the return of the levered strategy is determined by the return of the source portfolio, the leverage, and the financing cost associated with the leverage. By definition, leverage is constant; there is no trade and hence no trading costs; and there is no compounding to take into account.

Now consider a simple two-period model. Assume that the source portfolio earns a 10% arithmetic return in period 1 and a -10% arithmetic return in period 2. We invest $100.00 which is worth $110.00 at the end of period 1 and $99.00 at the end of period 2, as shown in Table 1. The average of the arithmetic return over the two periods is zero, but the cumulative return of the source portfolio over the two periods is:

$$\frac{99 - 100}{100} = -0.01 = -1.00\%$$

The average arithmetic return of the source portfolio return must be corrected for compounding. As noted by Booth and Fama (1992) and discussed in Appendix D, the correction subtracts half the variance of arithmetic return each period; we call this corection the variance drag. Note that the variance of the arithmetic returns is:

$$\frac{(0.1 - 0)^2 + (-0.1 - 0)^2}{2} = .01 = 1.00\%$$

If we subtract half the variance from the arithmetic return each period, we get a total return of:

$$0.1 - \frac{.01}{2} + (-0.1) - \frac{.01}{2} = -0.01 = -1.00\%$$

which matches the actual cumulative return over the two periods.

Table 1: Strategies in the Two-Period Example

Time	Source Return r^S	Assets A (\$)	Debt D (\$)	Strategy Value $A - D$ (\$)
Source Portfolio:				
Beginning of 1		100.00	0.00	100.00
End of 1	10%	110.00	0.00	110.00
End of 2	-10%	99.00	0.00	99.00
Fixed Leverage Strategy:				
Beginning of 1		250.00	150.00	100.00
End of 1	10%	275.00	150.00	125.00
End of 1$'$		312.50	187.50	125.00
End of 2	-10%	281.25	187.50	93.75
Dynamically Levered Strategy:				
Beginning of 1		200.00	100.00	100.00
End of 1	10%	220.00	100.00	120.00
End of 1$'$		360.00	240.00	120.00
End of 2	-10%	324.00	240.00	84.00

Table 1: Calculation of return of the source portfolio and levered strategies in the two-period model. The rows with time End of 1 represent the levered strategy prior to rebalancing, while the rows with time End of 1$'$ represents the levered strategy after rebalancing.

Now, consider a levered strategy. Suppose for simplicity that leverage can be financed at the risk-free rate, which happens to be zero. We initially invest \$100.00. Suppose that we target a fixed volatility of 12% per period, and our estimate of the source portfolio volatility is 6% at the beginning of the first period and 4% at the beginning of the second period. Thus, we choose leverage $\lambda_1 = \frac{12}{6} = 2$ in the first period and $\lambda_2 = \frac{12}{4} = 3$ in the second period.

If we try to squeeze this into a one-period analysis, we might be tempted to assume the return will be similar to that of a strategy with fixed leverage $\bar{\lambda} = \frac{\lambda_1 + \lambda_2}{2} = 2.5$, since this is the average leverage over the two periods. Thus, we might expect to earn about 2.5 times the average arithmetic return of the source portfolio, or $2.5 \times 0.00\% = 0.00\%$. If we are a little more sophisticated and take compounding into account, we might expect to earn $2.5 \times -1.00\% = -2.50\%$. However, both these answers are wrong, even for the case of fixed leverage, and they are particularly wrong for the case of dynamic leverage.

Consider a fixed leverage strategy that uses leverage 2.5 in both periods. As noted in Table 1, we hold assets of \$250.00 in the source portfolio (financed by our \$100.00 and \$150.00 in debt) at the beginning of the first period. At the end of the first period, our assets are worth \$275.00, and our debt is \$150.00, so the value of the levered strategy is

$125.00. Even though we want to maintain fixed leverage, we need to rebalance. We hold assets of $125.00 \times 2.5 = 312.50 in the source portfolio. We must borrow, increasing our debt to $312.50-125.00=$187.50 to finance the position. At the end of the second period, our assets in the source portfolio are worth $312.50 \times 0.9 = 281.25, so the value of the levered strategy is $281.25 - 187.50 = 93.75; our cumulative return over the two periods is $\frac{93.75-100}{100} = -0.625$, a loss of 6.25%. The variance of levered strategy return is:

$$\frac{(.25 - 0)^2 + (-0.25 - 0)^2}{2} = .0625 = 6.25\%$$

The correction for compounding (variance drag) is $\frac{6.25\%}{2}$ per period; over two periods, this gives -6.25%, exactly matching the realized return. Note that the variance drag is *quadratic* in leverage, so constant leverage of 2.5 increases the variance drag by a factor of 6.25.

With our more sophisticated understanding of the quadratic nature of the variance drag, we might expect the dynamically levered strategy to have a cumulative return of about -6.25%. However, that answer is also wrong. As shown in Table 1, the dynamically levered strategy holds assets of $200.00 in the source portfolio at the beginning of the first period, financed by our initial $100.00 and debt of $100.00. At the end of the first period, the assets are worth a total of $220.00, the debt is still $100.00, so the value of the levered strategy is $120.00. We rebalance to achieve the prescribed leverage. Since the levered strategy calls for leverage $\lambda_2 = 3$, we borrow an additional $140.00 for total debt of $240.00, and hold assets of $360.00 in the source portfolio. We incur trading costs, which for simplicity we assume to be zero. At the end of the second period, our shares of the source portfolio are worth $360.00 \times 0.9 = 324.00; since we owe $240.00, so our equity is $84.00. The cumulative return to the levered strategy over the two periods is:

$$\frac{84 - 100}{100} = -.16 = -16.00\%$$

Rather than breaking even, or losing 2.5%, or losing 6.25% as we expected from our single-period intuition, we have lost 16%.

We went wrong because we ignored the *covariance* between leverage λ and source portfolio return $r^{\mathbf{S}}$:

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\begin{aligned}
\text{cov}(\lambda, r^{\mathbf{S}}) &= \frac{(\lambda_1 - \bar{\lambda}) \times (r_1^{\mathbf{S}} - 0) + (\lambda_2 - \bar{\lambda}) \times (r_2^{\mathbf{S}} - 0)}{2} \\
&= \frac{(2 - 2.5)(.1 - 0) + (3 - 2.5)(-0.1 - 0)}{2} \\
&= \frac{-0.05 - 0.05}{2} \\
&= -0.05 = -5.00\%
\end{aligned}
$$

The covariance term reduces return by 5.00% *each period*, producing a return of -10.00%. The arithmetic return of the dynamically levered strategy is 0.2 in the first

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period and -0.3 in the second periods, so the variance of the return of the levered strategy is:

$$\frac{(.2-(-0.05))^2+(-0.3-(-0.5))^2}{2}=\frac{(0.25)^2+(-0.25)^2}{2}=0.0625=6.25\%$$

The variance drag for the dynamically levered strategy is the same as for the fixed leverage strategy:[1] half the variance, or 3.125%, each period. Combining the covariance and variance, we get a return of -8.125%, per period, suggesting a cumulative loss of 16.25%, close to the cumulative actual return of -16.00% over the combination of the two periods.

As this example indicates, the covariance term can make a big difference over a few periods. One might be tempted to think the covariance term would wash out over time. If that were true, the covariance term might not be particularly important. Strikingly, we found that the covariance term made a *substantial difference* over a *very long horizon.* Our empirical examples include as source portfolios risk parity (with asset classes consisting of US stocks and US Treasury bonds) and Treasury bonds alone, with two different types of volatility targeting and two different volatility targets. In all of our examples, the covariance term turned out to be negative, subtracting between 0.64% and 4.23% from annualized return over an 84-year horizon. Consequently, the Sharpe ratios of volatility-targeting strategies were diminished relative to their source portfolios and fixed leverage benchmarks.

2   Synopsis of Theoretical Contributions and Empirical Findings

Hedge funds, real estate investment trusts, and many other investment vehicles routinely use leverage. Even among the most conservative and highly regulated investors such as US public pension funds, the use of levered investment strategies is widespread and growing.[2] In the period since the financial crisis, strategies such as risk parity that explicitly lever holdings of publicly traded securities have emerged as candidates for these investment portfolios.[3]

In the single-period Capital Asset Pricing Model (CAPM), the market portfolio is the unique portfolio of risky assets that maximizes the Sharpe ratio. Leverage serves only as a means to travel along the efficient frontier. Both excess return and volatility scale linearly with leverage, and a rational investor will lever or de-lever the market portfolio in accordance with his or her risk tolerance.

Empirically, certain low-volatility portfolios have exhibited higher Sharpe ratios than did the market portfolio,[4] which suggests that levering a low-volatility source portfolio

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[1]This is true in this specific example, but is not true in general.
[2]See, for example, Kozlowski (2013).
[3]Sullivan (2010) discusses the risks that a pension fund incurs by employing a levered strategy.
[4]See, for example, Anderson et al. (2012).

could deliver an attractive risk-return tradeoff. However, market frictions such as the difference between borrowing and lending rates, and the correlations that arise in multi-period models make the relationship between the realized return of a levered strategy and the Sharpe ratio of its source portfolio both nuanced and complex. Levered strategies tend to have substantially higher transaction costs[5] than do traditional strategies.[6]

We develop an exact performance attribution for levered strategies that takes market frictions into account. Specifically, we show that there are five important elements to cumulative return. The first element is the return to the fully invested portfolio to be levered, which we call the *source portfolio*. The second element is the expected return to the source in excess of the borrowing rate, amplified by leverage minus one. We call the sum of these terms the *magnified source return*, and it represents the performance of a levered strategy in an idealized world.

In the real world, the magnified source return is enhanced or diminished by the covariance between leverage and excess borrowing return, which is the third element of cumulative return of a levered strategy. Empirically, the covariance term turned out to be unstable at medium horizons of three to five years. Looking back, this made certain levered strategies appear particularly appealing at some times and particularly disappointing at other times. Viewed prospectively, it added considerable noise to medium horizon returns. The fourth and fifth elements, the cost of trading and the variance drag, are familiar to many investors. We penalized trading according to a linear model and we estimated the variance drag, which is effectively the difference between arithmetic and geometric return, using a formula that is adapted from Booth and Fama (1992).

Section 3 provides the foundation for our performance attribution, which is derived in Section 3.1. In Section 3.2, we illustrate the performance attribution in the context of a particular risk parity strategy, $\text{UVT}_{60/40}$, which targeted a fixed volatility equal to the realized volatility of a 60/40 fixed mix over our 84-year sample period, 1929–2012. As shown in Table 2, all five terms in the performance attribution contributed materially to the cumulative return of $\text{UVT}_{60/40}$. For example, the covariance term subtracted an average of 1.84% per year from the expected arithmetic return of the magnified source portfolio.

Section 4 discusses the assumptions we made about historical borrowing and trading costs and their impact on performance comparisons.

Our performance attribution facilitates a comparison between a levered strategy and a variety of benchmarks, which are explored in Section 5. The benchmarks fall into two classes. The first consists of fully invested portfolios, while the second consists of portfolios

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[5]Investment returns are often reported gross of fees and transaction costs. That practice may be reasonable in comparing strategies with roughly equal fees or transaction costs, but it is inappropriate when comparing strategies with materially different fees or transaction costs.

[6]By traditional strategies, we mean the strategies that have typically been employed over the last 50 years by pension funds and endowments. These strategies invest, without leverage, in a relatively fixed allocation among asset classes.

that use fixed leverage. For example, we compare $UVT_{60/40}$ to its source portfolio and its volatility target, a 60/40 fixed mix. The comparison of a levered strategy to fully invested benchmarks is important since there would be no rational reason to invest in a levered strategy if it underperformed these benchmarks. However the comparison is clouded by the fact that backtests of levered strategies rely on assumptions about historical financing costs, while backtests of unlevered strategies do not.[7]

By contrast, comparisons among backtests of different types of levered strategies are on firmer ground: even if there are errors in the assumptions about financing costs, they affect all the strategies under consideration in similar ways. We introduce two fixed-leverage strategies in Section 5.2. The first, $FLT_{60/40,\lambda}$ had constant leverage equal to the average leverage of $UVT_{60/40}$; the second, $FLT_{60/40,\sigma}$, had constant leverage and had volatility equal to the volatility of $UVT_{60/40}$. In our backtests, the fixed leverage strategies outperformed $UVT_{60/40}$ as well as a conditional volatility-targeting risk parity strategy, $CVT_{60/40}$, which is also introduced in Section 5.2. The volatility-targeting strategies had lower Sharpe ratios than the corresponding fixed-leverage strategies, which had lower Sharpe ratios than the underlying source portfolios. Section 5.3 discusses how the levered strategies UVT, CVT, and FLT responded to changes in market conditions; in particular, with $\lambda > 1$, these turned out to be momentum strategies.

$CVT_{60/40}$ matched the contemporaneous volatility of the fixed-mix 60/40, rather than its unconditional volatility over a long horizon. An advantage of $CVT_{60/40}$ over the other strategies is that is investable: perfect foresight is not required to rebalance the strategy each month. On the other hand, UVT and FLT strategies can be set by choosing a fixed volatility or leverage that is in the ball park of the expected future volatility of the target. This raises the question of sensitivity to parameters: if we set a UVT target volatility with an *intent* to match the volatility of a given strategy, such as the value-weighted market or 60/40, how close will the performance of the strategy we implement be to the performance of the strategy we intended to implement? We do not seriously address this question here, but a hint about its complexity and depth is in Section 5.4, which looks at the impact of the target volatility on strategy performance. These four risk parity strategies lever a common source portfolio, so it is straightforward to compare the return attributions of the strategies. The details are in Table 6, which shows, for example, that the covariance drag in $UVT_{60/40}$ was substantially larger than in $CVT_{60/40}$, and the difference in the covariance drags of UVT_{MKT} and CVT_{MKT} was even more pronounced. The high magnitude of the covariance drag and its sensitivity to the volatility target in UVT, came from both a high volatility of leverage and high sensitivity of the volatility of leverage to the UVT volatility target, compared to CVT.

In Section 5.5, we look beyond risk parity by considering a US government bond index levered to the volatility of US equities. The results are qualitatively similar although they

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[7]To the extent that levered strategies exhibited higher turnover than fully invested strategies, their returns may have been more sensitive to assumptions about historical trading costs. In our empirical results, financing costs had a significantly greater impact than did trading costs.

are more dramatic since the volatility of the source portfolio is lower in this example than in the others, while the target volatility is higher. The results are in Table 7. For example, the covariance term in UVTB$_{\text{STOCKS}}$ subtracted 4.23% per year from strategy performance.

Section 6 revisits the covariance term from the viewpoint of volatility-targeting. It demonstrates that the covariance term is still present from the volatility-matching perspective, and demonstrates that fixed-volatility targeting is a form of unintended market-timing, whereas fixed leverage is not.

In all of the volatility-targeting strategies we considered, the covariance term in the UVT strategies was negative over our 84-year data period. We note that the UVT covariance term was positive over many three-to-five year periods, and some periods lasting two to four decades.

Section 7 summarizes our main conclusions.

We also include a number of appendices that support our main narrative. Appendix A provides a detailed overview of the literature on low-risk investing and leverage. Appendix B describes the data in enough detail to allow researchers to replicate our results. Appendix C describes our linear trading model. Appendix D derives our approximation of geometric return from arithmetic return. As illustrated in our empirical examples, this approximation has a high degree of accuracy in practical situations. Appendix E presents a table with the formulas and corresponding words for the elements of our performance attribution.

3 The Impact of Leverage on the Return to an Investment Strategy

Leverage magnifies return, but that is only one facet of the impact that leverage has on an investment strategy. Leverage requires financing and exacerbates turnover, thereby incurring transaction costs. It amplifies the variance drag on cumulative return due to compounding. When leverage is dynamic, it can add substantial noise to strategy return. We provide an exact attribution of the cumulative return to a levered strategy that quantifies these effects.

A *levered strategy* is built from a fully invested *source portfolio* of risky assets, presumably chosen for its desirable risk-adjusted returns, and a *leverage rule*.[8]

An investor has a certain amount of capital, L. The investor chooses a leverage ratio λ, borrows $(\lambda - 1)L$, and invests λL in the source portfolio.[9]

In what follows, we assume $\lambda > 1$.

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[8]The source portfolio can be long-short in the risky assets. It must, however, have a non-zero value, so that returns can be calculated. Since we want to model leverage explicitly, we do not allow the source portfolio to contain a long or short position in a riskless asset, such as T-bills, the money market account, or commercial paper.

[9]Leverage may be achieved through explicit borrowing. It may also be achieved through the use of

3.1 Attribution of Arithmetic and Geometric Return

The relationship between the single-period return to a levered portfolio, $r^{\mathbf{L}}$, and to its source portfolio, $r^{\mathbf{S}}$, is given by:

$$r^{\mathbf{L}} = \lambda r^{\mathbf{S}} - (\lambda - 1) r^b, \tag{1}$$

where the borrowing rate, r^b, is greater than or equal to the risk-free rate r^f. Note that the excess return is given by:

$$r^{\mathbf{L}} - r^f = \lambda r^{\mathbf{S}} - (\lambda - 1) r^b - r^f$$
$$= \lambda \left(r^{\mathbf{S}} - r^f \right) - (\lambda - 1) \left(r^b - r^f \right) \tag{2}$$

Excess return and volatility scale linearly in λ for $\lambda \geq 0$ if and only if $r^b = r^f$; in that case, the situation is essentially the same as the single-period CAPM, except that the source portfolio need not be the market portfolio.

When $r^b > r^f$, volatility still scales linearly in $\lambda \geq 0$ but Formula (2) indicates that excess return scales sublinearly; as a consequence, the Sharpe ratio is a *declining* function of λ. Note that the excess borrowing return of the levered strategy is:

$$r^{\mathbf{L}} - r^b = \lambda \left(r^{\mathbf{S}} - r^b \right) \tag{3}$$

It is the excess *borrowing* return and volatility that scale linearly in leverage, for $\lambda \geq 1$. The bar for leverage to have a positive impact on return has gotten higher: the excess borrowing return, $r^{\mathbf{S}} - r^b$, must be positive.

The expected return to a levered strategy is estimated by rewriting Formula (3) as:

$$r^{\mathbf{L}} = r^{\mathbf{S}} + (\lambda - 1) \left(r^{\mathbf{S}} - r^b \right) \tag{4}$$

and taking the expectation over multiple periods:

$$\begin{aligned} E\left[r^{\mathbf{L}} \right] &= E\left[r^{\mathbf{S}} \right] + E\left[(\lambda - 1) \left(r^{\mathbf{S}} - r^b \right) \right] \\ &= E\left[r^{\mathbf{S}} \right] + E\left[\lambda - 1 \right] E\left[r^{\mathbf{S}} - r^b \right] + \mathrm{cov}(\lambda, r^{\mathbf{S}} - r^b) \end{aligned} \tag{5}$$

We use the term *magnified source return* to denote the sum of the first two terms on the right side of Formula (5). That formula shows that the expected return to a levered strategy is equal to the magnified source return plus a covariance correction. We find empirically that, even when the correlation between leverage and excess borrowing return is quite small, the covariance correction can be substantial in relation to the magnified source return.

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derivative contracts, such as futures. In these derivative contracts, the borrowing cost is implicit rather than explicit, but it is real and is typically at a rate higher than the T-Bill rate. For example, Naranjo (2009) finds that the implicit borrowing cost using futures is approximately the applicable LIBOR rate, applied to the notional value of the futures contract.

We can interpret the expectation and covariance in Formula (5) in two ways. Prospectively, they represent the expectation and covariance under the true probability distribution. Retrospectively, they represent the realized mean and realized covariance of the returns.[10]

Also important over multiple periods is the cost of trading, which imposes a drag $r^{\mathbf{TC}}$ on any strategy: To take account of this effect, we extend Formula (5):

$$E\left[r^{\mathbf{L}}\right] = E\left[r^{\mathbf{S}}\right] + E\left[\lambda - 1\right] E\left[r^{\mathbf{S}} - r^b\right] + \text{cov}(\lambda, r^{\mathbf{S}} - r^b) - E\left[r^{\mathbf{TC}}\right]$$
$$= E\left[r^{\mathbf{S}}\right] + E\left[\lambda - 1\right] E\left[r^{\mathbf{S}} - r^b\right] + \text{cov}(\lambda, r^{\mathbf{S}} - r^b) - \left(E\left[r^{\mathbf{TCS}}\right] + E\left[r^{\mathbf{TCL}}\right]\right) \quad (6)$$

where $r^{\mathbf{TC}}$ is expressed as a sum of trading costs due to turnover in the source portfolio and trading costs due to leverage-induced turnover:

$$r^{\mathbf{TC}} = r^{\mathbf{TCS}} + r^{\mathbf{TCL}}.$$

Estimates of $r^{\mathbf{TC}}$ and its components rely on assumptions about the relationship between turnover and trading cost. We assumed that cost depended linearly on the dollar value that turned over, and we used Formulas (15) and (16) to estimate $r^{\mathbf{TC}}$ in our empirical studies. More information is in Appendix C.

Formula (6) is based on arithmetic expected return, which does not correctly account for compounding. The correction for compounding imposes a variance drag on cumulative return that affects strategies differentially; for any given source portfolio, the variance drag is quadratic in leverage. If a levered strategy has high volatility, the variance drag may be substantial.

If we have monthly returns for months $t = 0, 1 \ldots, T - 1$ the realized geometric average of the monthly returns is:

$$G[r] = \left(\prod_{t=0}^{T-1}(1 + r_t)\right)^{1/T} - 1 \quad (7)$$

where r_t is the arithmetic return in month t. Given two strategies, the one with the higher realized geometric average will have higher realized cumulative return. In Appendix D, we show that the following holds to a high degree of approximation:[11]

$$G\left[r\right] \quad \sim \quad (1 + E\left[r\right]) e^{-\frac{\text{var}(r)}{2}} - 1 \quad (8)$$

Note that the correction depends only on the realized variance of return.[12] Booth and Fama (1992) provide a correction for compounding based on continuously compounded

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[10]Note that we take the realized covariance, obtained by dividing by the number of dates, rather than the realized sample covariance, which would be obtained by dividing by one less than the number of dates. We use the realized covariance because it makes Formula (5) true.

[11]The magnitude of the error is estimated following Formula (20). Note that G and E denote realizations of the geometric and average arithmetic return, respectively. The term $\text{var}(r)$ denotes the realized variance of r, rather than the realized sample variance.

[12]In an earlier version of this paper, we indicated, incorrectly, that both the level and the variability of volatility determine the magnitude of the variance drag.

return; our correction for the geometric average of monthly returns in Formula (8) is slightly simpler.

Thus, in comparing the realized returns of strategies, the magnified source return of the levered strategy must be adjusted for three factors that arise only in the multi-period setting: the covariance correction, the variance drag, and trading costs.[13]

3.2 Empirical Example: Performance Attribution of a Levered Risk Parity Strategy

We demonstrate the utility of the performance attribution detailed above in the context of $UVT_{60/40}$, a risk parity strategy that was rebalanced monthly and levered to an unconditional volatility target equal to the realized volatility, 11.59%, of the 60/40 fixed-mix between January 1929 and December 2012.[14] The source portfolio was unlevered risk parity based on two asset classes, US Equity and US Treasury Bonds. Foresight was required in order to set this target: the volatility of the 60/40 strategy was not known until the end of the period.[15]

Figure 1 shows the magnified source return and the realized cumulative return to $UVT_{60/40}$, as well as the realized cumulative return to its source portfolio (fully invested risk parity) and target (60/40 fixed mix). All computations assumed that leverage is financed at the 3-month Eurodollar deposit rate. The realized cumulative returns were based on the additional assumption that trading is penalized according to the linear model described in Appendix C, and took into account the covariance correction and variance drag on cumulative return. The magnified source return of $UVT_{60/40}$ easily beat the cumulative return of both the source and the target; however, the realized cumulative return of $UVT_{60/40}$ was well below the realized cumulative return of the 60/40 target portfolio (with essentially equal volatility (11.58%)) and only slightly better than *unlevered* risk parity source portfolio, which had much lower volatility (4.20%).[16]

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[13]Note that the source and target portfolios may incur their own trading costs, as well as benefit from volatility pumping. The performance attribution of Formula (6) uses the source return and magnified source return, *gross* of trading costs. When we report historical arithmetic returns to the source and target portfolio, we report these net of trading costs, and inclusive of any benefit from volatility pumping. When we report cumulative returns to the source and target portfolios, we report these net of the variance drag.

[14]The leverage was chosen so that the volatility, gross of trading costs, was exactly 11.59%. When trading costs were taken into account, the realized volatility was slightly lower: 11.54%. $UVT_{60/40}$ was constructed in effectively the same way as the levered risk parity strategy in Asness et al. (2012), with one main difference. They levered risk parity to match the volatility of the market, which had higher volatility than 60/40. In Section 5.4, we consider risk parity levered to the volatility of the market.

[15]The sensitivity of strategy performance to the volatility target is discussed in Section 5.4.

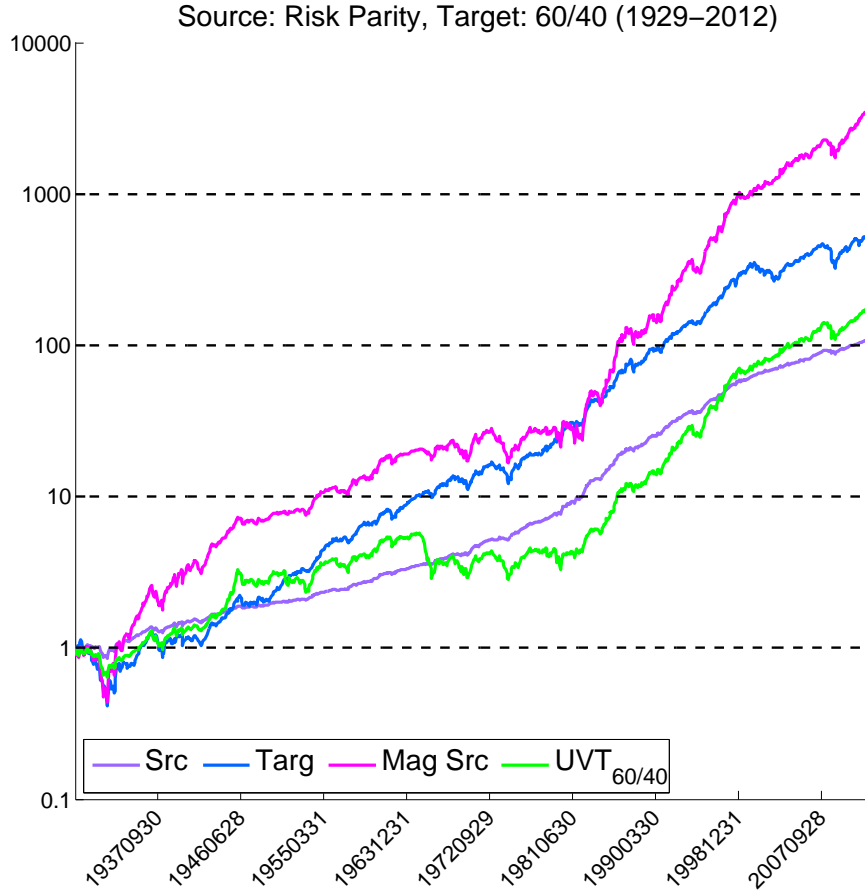[16]The volatilities are reported in Table 3.

Figure 1: Magnified source return (in magenta) and realized cumulative return (in light green) for $UVT_{60/40}$ (risk parity unconditionally levered to a target volatility of 11.59%) over the period 1929–2012. For comparison, we also plot the realized cumulative return of the volatility target (60/40 fixed mix, in blue) and the source (fully invested risk parity, in lavender). Magnified source return is an idealized return that cannot be achieved in practice; the curve depicts what we would earn if we achieved a geometric return equal to the arithmetic magnified source return.

The return decomposition Formulas (6) and (8) provide a framework for analyzing the performance of $UVT_{60/40}$. Table 2 provides the required information. Consider first the magnified source return. The source portfolio had an annualized arithmetic return of 5.75% gross of trading costs.[17] Leverage added an extra 3.97% to annualized return from the magnification term, the average excess borrowing return to the source portfolio multiplied by average leverage minus one. The annualized magnified source return was thus 9.72%. However, the covariance between leverage and excess borrowing return reduced the annualized return by 1.84%, trading costs by 96 basis points, and variance drag by a

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[17]Trading costs subtracted only 7 basis points per year from the source return.

further 48 basis points. Together, these three effects ate up 3.28% of the 3.97%, or 82.6%, of the contribution of leverage to the magnified source return.

Table 2: Performance Attribution

Sample Period: 1929-2012 Source: Risk Parity, Target: 60/40 r^b = 3M-EDR, with trading costs		$\text{UVT}_{60/40}$
Total Source Return (gross of trading costs)		**5.75**
Leverage	2.66	
Excess Borrowing Return	1.49	
Levered Excess Borrowing Return		3.97
Magnified Source Return		**9.72**
Volatility of Leverage	7.7212	
Volatility of Excess Borrowing Return	4.2219	
Correlation(Leverage,Excess Borrowing Return)	-0.0566	
Covariance(Leverage,Excess Borrowing Return)		-1.84
Source Trading Costs		-0.07
Leverage-Induced Trading Costs		-0.96
Total Levered Return (arithmetic)		**6.85**
Compounded Arithmetic Return (gross)	1.0707	
Variance Correction	0.9934	
Variance Drag		-0.48
Approximation Error		0.00
Total Levered Return (geometric)		**6.37**

Table 2: Performance attribution of the realized geometric return of the levered strategy $\text{UVT}_{60/40}$ in terms of its source portfolio, risk parity, over the period January 1929–December 2012. The performance attribution was based on Formulas (6) and (8). Borrowing was at the Eurodollar deposit rate and trading costs were based on the linear model in Appendix C. Arithmetic returns were estimated from monthly data and annualized by multiplication by 12; they are displayed in percent. Geometric returns are also displayed in percent and were annualized by $(1 + G[r])^{12} - 1$. Formulas corresponding to the words in the performance attribution are presented in Table 9.

4 Assumptions about Transaction Costs and Their Impact on Empirical Results

The return calculations in our empirical examples relied on assumptions about transaction costs over our study period, 1929–2012. Comparisons between levered and unlevered strategies were sensitive to these assumptions, but comparisons between strategies that

were comparably levered were much less sensitive to them. For transparency, we include the details of our assumptions about transaction costs in Appendices B and C. Here, we explain some of the reasoning that led to the choices we made, and we discuss the impact of our choices on the results.

One guideline is that trading became less expensive over time during the study period, so we assessed a greater cost to turnover at the beginning of the period than the end. Specifically, we assumed that the portfolio was rebalanced monthly[18] and that trading cost 1% of the dollar amount of a trade between 1929 and 1955, .5% between 1956 and 1971, and .1% between 1972 and 2012. Since turnover tended to be higher in a levered strategy than in an unlevered strategy, higher trading costs tended to do more damage to a levered strategy than to an unlevered strategy.

As a borrowing rate, we used the 3-month Eurodollar deposit rate, for which we had data back to the beginning of 1971. Prior to 1971, we used the 3-month T-bill rate plus a spread of 60 basis points, which was 40 basis points less than the average spread between the Eurodollar deposit rate and the T-bill rate between 1971 and 2012. This choice improved the performance of our levered strategies relative to what they would have been had we used the average spread. Of course, a lower borrowing rate would have further improved the performance of the levered strategies.[19] Since the levered strategies involved borrowing and the unlevered strategies did not, there were more assumptions underlying the empirical results for levered strategies than for unlevered strategies. As a consequence, our uncertainty about results for levered strategies was greater than for unlevered strategies.

It would, of course, have been possible to include empirical results based on a wider range of assumptions about transaction costs. However, that would have been misleading since it would have conveyed the impression that we had done a thorough study of the issue. We did not. We chose a streamlined approach of providing examples based on single set of assumptions that are consistent with published literature and that rely on readily available data. The purpose of these examples is to illustrate the efficacy of our performance attribution framework. We encourage practitioners and scholars to apply our framework using their own estimates of trading and borrowing costs in order to evaluate strategies and to facilitate the decision to lever.

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[18]In practice, trading costs can be reduced by reducing the frequency or completeness of rebalancing, at the cost of introducing tracking error. Further, trading costs may be higher for some asset classes than for others. However, in our empirical examples, financing costs were more important than trading costs.

[19]We considered using 1-month rates, but that would have engendered a more complex extrapolation since the 1-month T-bill rate began only in 2001. Note that the difference between the 1-month and 3-month Eurodollar deposit rates averaged 20 basis points between 1971 and 2013. This was offset by the 40 basis points we subtracted in our extrapolation.

5 Benchmarks for a Levered Strategy

5.1 Fully Invested Benchmarks

Table 3 reports annualized arithmetic and geometric return, volatility and Sharpe ratio to $UVT_{60/40}$, its source, and its target. Because $UVT_{60/40}$ was levered, while the source and target were not, these comparisons were subject to uncertainty about historic financing and trading costs. $UVT_{60/40}$ had annualized geometric return only 63 basis points higher than the source portfolio, unlevered risk parity.[20] At the same time, the source portfolio had a much lower volatility (4.20%). As a result, $UVT_{60/40}$ had a Sharpe ratio of 0.29, compared to 0.52 for unlevered risk parity. Note that the high Sharpe ratio of unlevered risk parity was obtained at the cost of low expected return.

60/40 and $UVT_{60/40}$ had essentially equal volatilities. Under our assumptions on historic financing and trading costs, 60/40 delivered an annualized geometric return of 7.77% and a realized Sharpe ratio of 0.40, while the analogous figures for $UVT_{60/40}$ were 6.37% and 0.29. Investors who are considering an investment in risk parity or any levered strategy can populate Tables 2 and 3 with their forward-looking estimates of the components of strategy return. This analysis can inform the decision to invest in a levered strategy instead of the fully invested source or target portfolio.

5.2 Fixed Leverage and Conditional Leverage Benchmarks

In this section, we focus on comparisons of realized returns among levered strategies that were constructed in different ways. These comparisons were less sensitive to the assumptions on historical financing and trading costs. Like any volatility targeting strategy, $UVT_{60/40}$ was dynamically levered. However, as we saw in Section 3.2, the covariance between leverage and excess borrowing return diminished annualized arithmetic return by 1.84%. Deeper insight into this cost is provided in Table 2, which decomposes these covariances into products of correlation and standard deviations. Note that the magnitude of the correlation between leverage and excess borrowing return was small: -0.056. Figure 2 shows rolling 36-month estimates of the correlation between leverage and excess borrowing return, and indicates that the sign of the correlation flipped repeatedly at short horizons. At investment horizons of three to five years, the main effect of the covariance term appeared to be to add noise to the returns.

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[20]Note that the annualized geometric return of the source portfolio, 5.74%, slightly exceeded 5.68%, the annualized arithmetic return of the source portfolio, net of trading costs. This is an artifact of the annualization procedures for arithmetic and geometric return. The source portfolio had monthly arithmetic return of 47.3 basis points, net of transaction costs. The latter was annualized by multiplying by 12: $12 \times 0.473\% = 5.68\%$. Annualized geometric return takes into account compounding: $1.00473^{12} - 1 = 5.83\%$. The variance drag reduced this by 9 basis points to 5.74%. The variance drag on the source return was much smaller than the variance drag on the levered portfolios, because the source portfolio was so much less volatile and the variance drag is quadratic in volatility.

Table 3: Historical Performance

Sample Period: 1929-2012 Source: Risk Parity, Target: 60/40 r^b = 3M-EDR	Arithmetic Total Return	Geometric Total Return	Average Leverage	Volatility	Arithmetic Excess Return	Sharpe Ratio	Skewness	Excess Kurtosis
60/40	8.18	7.77	1.00	11.58	4.69	0.40	0.19	7.44
Risk Parity	5.68	5.74	1.00	4.20	2.20	0.52	0.05	4.92
$UVT_{60/40}$	6.85	6.37	3.66	11.54	3.37	0.29	-0.43	2.23

Table 3: Annualized arithmetic and geometric returns, volatility and Sharpe ratio, of $UVT_{60/40}$ (risk parity levered to an unconditional volatility target of 11.59%, the realized volatility of 60/40), the source portfolio (unlevered risk parity), and the volatility target (60/40) over the period 1929–2012. Arithmetic returns were estimated from monthly data and annualized by multiplication by 12; they are displayed in percent. Geometric returns are also displayed in percent and were annualized by $(1 + G[r])^{12} - 1$. Volatility was measured from monthly returns and annualized by multiplying by $\sqrt{12}$. Sharpe ratios were calculated using annualized excess return and annualized volatility.
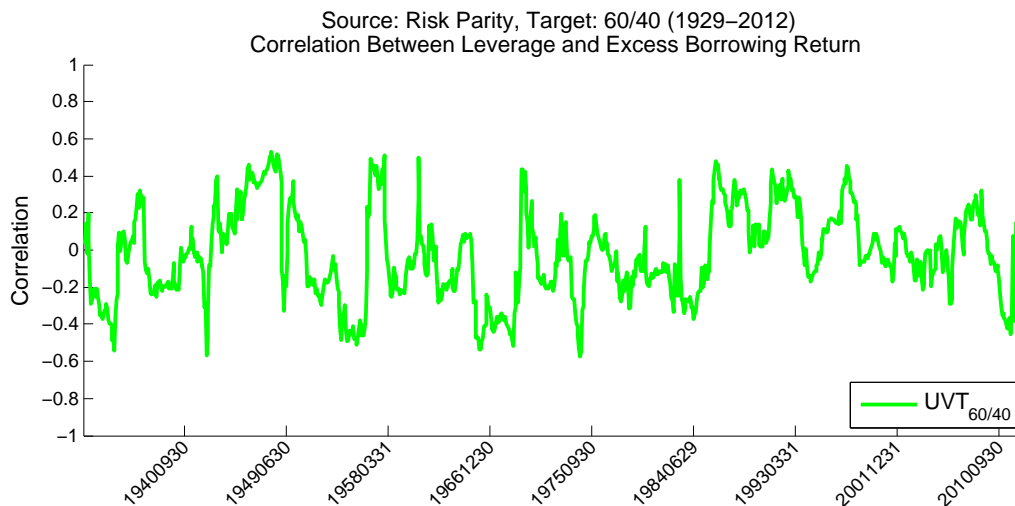
Figure 2: Correlation of excess borrowing return and leverage for $UVT_{60/40}$, risk parity levered to match the realized volatility of 60/40 fixed mix over the period 1929–2012. Correlation was computed from monthly data using a trailing 36-month window.

When leverage is fixed, the covariance between leverage and excess borrowing return must be zero. We consider two fixed leverage strategies: $FLT_{60/40,\lambda}$ matched the average leverage of $UVT_{60/40}$, but had higher volatility, while $FLT_{60/40,\sigma}$ matched the volatility of $UVT_{60/40}$ but had lower leverage.

Another alternative to UVT is a conditional volatility targeting strategy. $CVT_{60/40}$

levered fully invested risk parity so that the projected volatility (based on the previous 36 months returns) equalled the volatility of the target 60/40 over the previous 36 months.[21]

Table 4 provides performance attributions for $\text{UVT}_{60/40}$, $\text{FLT}_{60/40,\lambda}$, $\text{FLT}_{60/40,\sigma}$ and $\text{CVT}_{60/40}$. Note that each column of Table 4 is a version of Table 2 applied to one of our four levered strategies. All four levered strategies made use of the same source portfolio, and hence had the same source arithmetic return. Leverage contributed substantially and at roughly the same level to the magnified source return of $\text{UVT}_{60/40}$, $\text{FLT}_{60/40,\lambda}$ and $\text{CVT}_{60/40}$, since those three strategies had similar average leverage. The contribution to the return of $\text{FLT}_{60/40,\sigma}$ was significantly lower because that strategy had lower average leverage. The covariance term reduced the annualized arithmetic return of $\text{UVT}_{60/40}$ by 1.84%, but led to a much smaller reduction in the return of $\text{CVT}_{60/40}$ and, by design, had no effect on the return of the two FLT strategies. Trading costs reduced the return of $\text{UVT}_{60/40}$ and $\text{CVT}_{60/40}$ by about 95 basis points, but had a smaller effect on the two FLT strategies.[22] The variance drag reduced the geometric returns of $\text{UVT}_{60/40}$, $\text{FLT}_{60/40,\sigma}$ and $\text{CVT}_{60/40}$ by similar amounts, since these strategies had similar variances; the effect on $\text{FLT}_{60/40,\lambda}$ was greater as a result of its higher volatility. When all the effects were taken into account, the geometric returns of $\text{FLT}_{60/40,\lambda}$, $\text{FLT}_{60/40,\sigma}$ and $\text{CVT}_{60/40}$ exceeded the geometric return of $\text{UVT}_{60/40}$ by 192, 125 and 66 basis points, respectively.

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[21]$\text{CVT}_{60/40}$ was introduced in Anderson et al. (2012).

[22]As discussed in Section 5.3 below, even maintaining a fixed leverage requires trading. It is possible in principle that the trading needed to adjust leverage to meet a volatility target could offset some of the trading required to maintain fixed leverage, but this strikes us as unlikely in typical situations. Had we assumed lower trading costs, it would have narrowed the gap in trading costs among the strategies, but not changed the ranking of those costs.

Table 4: Performance Attribution

Sample Period: 1929-2012 Source: Risk Parity, Target: 60/40 r^b = 3M-EDR, with trading costs	$UVT_{60/40}$		$FLT_{60/40,\lambda}$		$FLT_{60/40,\sigma}$		$CVT_{60/40}$	
Total Source Return (gross of trading costs)		**5.75**		**5.75**		**5.75**		**5.75**
Leverage	2.66		2.69		1.75		2.31	
Excess Borrowing Return	1.49		1.49		1.49		1.49	
Levered Excess Borrowing Return		3.97		4.02		2.61		3.45
Magnified Source Return		**9.72**		**9.77**		**8.37**		**9.20**
Volatility of Leverage	7.7212		0.0000		0.0000		5.0791	
Volatility of Excess Borrowing Return	4.2219		4.2219		4.2219		4.2219	
Correlation(Leverage,Excess Borrowing Return)	-0.0566		0.0000		0.0000		-0.0299	
Covariance(Leverage,Excess Borrowing Return)		-1.84		0.00		0.00		-0.64
Source Trading Costs		-0.07		-0.07		-0.07		-0.07
Leverage-Induced Trading Costs		-0.96		-0.51		-0.27		-0.93
Total Levered Return (arithmetic)		**6.85**		**9.19**		**8.03**		**7.56**
Compounded Arithmetic Return (gross)	1.0707		1.0959		1.0833		1.0783	
Variance Correction	0.9934		0.9881		0.9934		0.9926	
Variance Drag		-0.48		-0.91		-0.41		-0.53
Approximation Error		0.00		0.01		0.01		0.00
Total Levered Return (geometric)		**6.37**		**8.29**		**7.62**		**7.03**

Table 4: Performance attribution of the realized geometric return of the levered strategies $UVT_{60/40}$, $FLT_{60/40,\lambda}$, $FLT_{60/40,\sigma}$, and $CVT_{60/40}$ in terms of their common source portfolio, risk parity, over the period January 1929–December 2012. $FLT_{60/40,\lambda}$ had constant leverage 3.69, matching the average leverage of $UVT_{60/40}$, while $FLT_{60/40,\sigma}$ had constant leverage 2.75, chosen to match the volatility of $UVT_{60/40}$. The performance attribution was based on Formulas (6) and (8). Borrowing was at the Eurodollar deposit rate and trading costs were based on the linear model in Appendix C. Arithmetic returns were estimated from monthly data and annualized by multiplication by 12; they are displayed in percent. Geometric returns are also displayed in percent and were annualized by $(1 + G[r])^{12} - 1$. Formulas corresponding to the words in the performance attribution are presented in Table 9.

5.3 Attributes of Levered Strategies

The parameters of the UVT and two FLT levered strategies were set with foresight. The dynamically levered strategy $UVT_{60/40}$ was based on the realized volatility of a 60/40 fixed mix between January 1929 and December 2012. That volatility was known only at period end even though it was used to make leverage decisions throughout the period. The $FLT_{60/40,\lambda}$ leverage was set to match the average leverage of $UVT_{60/40}$ and the $FLT_{60/40,\sigma}$ leverage was set so that the volatility matched the volatility of $UVT_{60/40}$.

$CVT_{60/40}$, introduced in Section 5.2, did not rely on future information to set leverage.[23] As a result, its realized volatility failed to match the realized volatility of the target. At each monthly rebalancing, $CVT_{60/40}$ was levered to match the volatility of the 60/40 fixed mix; both volatilities were estimated using a 36-month rolling window.

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[23]The foresight in the definitions of UVT and the two FLT strategies allowed them to exactly match their volatility or leverage targets, gross of trading costs. Since $CVT_{60/40}$ did not rely on foresight, it could not exactly match the realized target volatility, gross of trading costs. Both UVT and $CVT_{60/40}$ volatility and FLT leverage were further affected by trading costs.

All else equal, $\text{UVT}_{60/40}$, $\text{FLT}_{60/40,\lambda}$, $\text{FLT}_{60/40,\sigma}$ and $\text{CVT}_{60/40}$ called for additional investment in the source portfolio when its price rose. A decline in the value of the source portfolio reduced the net value of the levered portfolio, while keeping the amount borrowed constant; leverage had increased, and rebalancing required selling the source portfolio to return to leverage λ. Similarly, an increase in the value of the source portfolio resulted in taking on more debt and using the proceeds to buy more of the source portfolio. In this sense, the UVT, FLT and CVT strategies with $\lambda > 1$ were momentum strategies. UVT, FLT and CVT strategies responded differently to changes in asset volatility; see Table 5.

Table 5: Strategy Responses to Changes in Market Conditions

| | Response: | | |
Trigger	FLT	UVT	CVT
Increase in Target Volatility	no change	no change	↑ leverage
Increase in Source Volatility	no change	↓ leverage	↓ leverage
Increase in Price of Source	buy source	buy source	buy source

Table 5: Responses of levered strategies to changes in market conditions.

5.4 Changing the Volatility Target

In this section, we explore the relationship between UVT and CVT strategies, and in particular their sensitivity to the volatility target. In addition to 60/40, we used the Market Portfolio (i.e. the value-weighted portfolio of stocks and bonds, which has a higher volatility than 60/40) as the volatility target. UVT_{MKT} and CVT_{MKT} denote unconditionally levered and conditionally levered risk parity strategies with the market as the volatility target. Return comparisons of UVT_{MKT} to CVT_{MKT} and of $\text{UVT}_{60/40}$ to $\text{CVT}_{60/40}$ were not sensitive to our assumptions on historical financing and trading costs, while the comparisons of UVT_{MKT} to $\text{UVT}_{60/40}$ and of CVT_{MKT} to $\text{CVT}_{60/40}$ were only slightly sensitive to those assumptions.

Each term in the return attribution of the UVT risk parity strategies was sensitive to the choice of MKT or 60/40 as the volatility target. By contrast, the magnified source returns, covariance terms and trading costs of CVT_{MKT} were quite similar to those of $\text{CVT}_{60/40}$; the only large difference between the two CVT strategies lay in the variance drag. This finding indicates that CVT strategies were more stable than UVT strategies.

The geometric returns of UVT_{MKT} (6.53%) and CVT_{MKT} (6.52%) were virtually tied, while $\text{CVT}_{60/40}$ outperformed $\text{UVT}_{60/40}$ by 63 basis points.[24]

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[24]These findings do not support the assertion by Asness et al. (2013) that CVT is an inherently inferior

Table 6: Performance Attribution

Sample Period: 1929-2012 Source: Risk Parity, Targets: VW Market, 60/40 r^b = 3M-EDR, with trading costs	UVT$_{\text{MKT}}$		UVT$_{60/40}$		CVT$_{\text{MKT}}$		CVT$_{60/40}$	
Total Source Return (gross of trading costs)		**5.75**		**5.75**		**5.75**		**5.75**
Leverage	3.71		2.66		2.58		2.31	
Excess Borrowing Return	1.49		1.49		1.49		1.49	
Levered Excess Borrowing Return		5.55		3.97		3.85		3.45
Magnified Source Return		**11.30**		**9.72**		**9.60**		**9.20**
Volatility of Leverage	9.9463		7.7212		5.3164		5.0791	
Volatility of Excess Borrowing Return	4.2219		4.2219		4.2219		4.2219	
Correlation(Leverage,Excess Borrowing Return)	-0.0566		-0.0566		-0.0321		-0.0299	
Covariance(Leverage,Excess Borrowing Return)		-2.37		-1.84		-0.72		-0.64
Source Trading Costs		-0.07		-0.07		-0.07		-0.07
Leverage-Induced Trading Costs		-1.40		-0.96		-1.13		-0.93
Total Levered Return (arithmetic)		**7.45**		**6.85**		**7.68**		**7.56**
Compounded Arithmetic Return (gross)	1.0771		1.0707		1.0796		1.0783	
Variance Correction	0.9891		0.9934		0.9872		0.9926	
Variance Drag		-0.92		-0.48		-1.11		-0.53
Approximation Error		0.00		0.00		-0.05		0.00
Total Levered Return (geometric)		**6.53**		**6.37**		**6.52**		**7.03**

Table 6: Performance attribution of the realized geometric return of the levered strategies UVT$_{\text{MKT}}$, UVT$_{60/40}$, CVT$_{\text{MKT}}$ and CVT$_{60/40}$ in terms of their common source portfolio, risk parity, over the period January 1929–December 2012. The performance attribution was based on Formulas (6) and (8). Borrowing was at the Eurodollar deposit rate and trading costs were based on the linear model in Appendix C. Arithmetic returns were estimated from monthly data and annualized by multiplication by 12; they are displayed in percent. Geometric returns are also displayed in percent and were annualized by $(1 + G[r])^{12} - 1$. Formulas corresponding to the words in the performance attribution are presented in Table 9.

5.5 Changing the Source Portfolio

Thus far, we have illustrated our performance attribution on a variety of risk parity strategies that share a common source portfolio, unlevered risk parity. That allowed us to isolate the impact of different leverage rules on performance.

In this section, we examine the impact of the source portfolio on performance: we consider strategies that levered an index of US government bonds to target the volatility of US equities. As in the previous examples, we consider both a dynamically levered volatility targeting strategy, UVTB$_{\text{STOCKS}}$, as well as fixed leverage benchmarks, FLTB$_{\text{STOCKS},\lambda}$ (with the same average leverage as UVTB$_{\text{STOCKS}}$) and FLTB$_{\text{STOCKS},\sigma}$ (with the same volatility as UVTB$_{\text{STOCKS}}$). The details, presented in Table 7, were qualitatively similar to what we saw for the risk parity strategies in Tables 4 and 6: an attractive magnified source return was diminished substantially by transaction costs for all levered strategies and by the covariance term for the dynamically levered strategy, UVTB$_{\text{STOCKS}}$. However, since the source portfolio had lower volatility than unlevered risk parity, and the target volatility was higher than that of 60/40 and the value-weighted market, leverage was

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implementation of risk parity, compared to UVT.

higher and the effects were more dramatic.

The covariance term for $\text{UVTB}_{\text{STOCKS}}$ was -4.23% per year, which imposed a larger drag on return than did the covariance terms (-1.84% and -2.73%) for $\text{UVT}_{60,40}$ and UVT_{MKT}. Despite the fact that the volatility target in $\text{UVTB}_{\text{STOCKS}}$ was *fixed*, the leverage was highly variable due to changes in the inverse of the volatility of the source portfolio of U.S. Treasury bonds.[25]

The correlation between leverage and excess borrowing return to the source portfolio was -.07. So as in the case of the dynamically levered risk parity strategies, a small correlation resulted in a large return drag. The geometric returns to $\text{FLTB}_{\text{STOCKS},\lambda}$ and $\text{FLTB}_{\text{STOCKS},\sigma}$ over our 84-year horizon were 5.93% and 6.94% per year. The geometric return to $\text{UVTB}_{\text{STOCKS}}$ over the same period was 1.7% per year.

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[25]See Section 6 for an analysis of the covariance term from the standpoint of volatility targeting. Had we made the unrealistic assumptions that financing was at the risk-free rate, and that trading costs were zero, the two FLT strategies would still have easily outperformed the UVT strategy.

Table 7: Performance Attribution

Sample Period: 1929-2012 Source: Bonds, Target: Stocks r^b = 3M-EDR, with trading costs	$\text{UVTB}_{\text{STOCKS}}$		$\text{FLTB}_{\text{STOCKS},\lambda}$		$\text{FLTB}_{\text{STOCKS},\sigma}$	
Total Source Return (gross of trading costs)		**5.08**		**5.08**		**5.08**
Leverage	6.43		6.49		4.80	
Excess Borrowing Return	0.82		0.82		0.82	
Levered Excess Borrowing Return		5.29		5.34		3.95
Magnified Source Return		**10.37**		**10.42**		**9.03**
Volatility of Leverage	17.4159		0.0000		0.0000	
Volatility of Excess Borrowing Return	3.2711		3.2711		3.2711	
Correlation(Leverage,Excess Borrowing Return)	-0.0742		0.0000		0.0000	
Covariance(Leverage,Excess Borrowing Return)		-4.23		0.00		0.00
Source Trading Costs		0.00		0.00		0.00
Leverage-Induced Trading Costs		-2.59		-1.62		-0.93
Total Levered Return (arithmetic)		**3.55**		**8.80**		**8.10**
Compounded Arithmetic Return (gross)	1.0361		1.0916		1.0841	
Variance Correction	0.9820		0.9707		0.9823	
Variance Drag		-1.80		-2.84		-1.61
Approximation Error		-0.05		-0.02		0.00
Total Levered Return (geometric)		**1.70**		**5.93**		**6.49**

Table 7: Performance attribution of the realized geometric return of the levered strategies $\text{UVTB}_{\text{STOCKS}}$, $\text{FLTB}_{\text{STOCKS},\lambda}$ and $\text{FLTB}_{\text{STOCKS},\sigma}$ in terms of their common source portfolio, U.S. Treasury bonds, over the period January 1929–December 2012. $\text{UVTB}_{\text{STOCKS}}$ was levered to the volatility of stocks (18.93%) over the period 1929–December 2012. $\text{FLTB}_{\text{STOCKS},\lambda}$ had fixed leverage 8.72, equal to the average leverage of $\text{UVTB}_{\text{STOCKS}}$; $\text{FLTB}_{\text{STOCKS},\sigma}$ had fixed leverage and the same volatility 22.47% as $\text{UVTB}_{\text{STOCKS}}$. The performance attribution was based on Formulas (6) and (8). Borrowing was at the Eurodollar deposit rate and trading costs were based on the linear model in Appendix C. Arithmetic returns were estimated from monthly data and annualized by multiplication by 12; they are displayed in percent. Geometric returns are also displayed in percent and were annualized by $(1 + G[r])^{12} - 1$. Formulas corresponding to the words in the performance attribution are presented in Table 9.

5.6 Historical Performance of the Various Levered and Fully Invested Strategies

Table 8 summarizes the historical performance of our source portfolios (unlevered risk parity and U.S. Treasury Bonds), volatility targets (fully invested 60/40, value-weighted market, and stocks) and the various levered strategies considered in this paper. Unlevered risk parity has the highest Sharpe ratio (0.52), followed closely by U.S. Treasury Bonds (0.49). However, both exhibited low volatility and low excess return, making them unattractive as asset allocations for most investors.[26] Levered strategies are attractive as an asset allocation only if the Sharpe ratio survives leverage.

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[26]Of course, bonds are often used as one asset class in an asset allocation, such as 60/40 or the value-weighted market portfolio. 60/40 has been widely used as an asset allocation, and risk parity has been proposed as an alternative asset allocation; see, for example, Asness et al. (2012).

As shown in Table 8, the Sharpe ratios of the levered strategies were all lower than the Sharpe ratios of their source portfolios. This highlights a fact that is well-known but often neglected: outside of an idealized setting, the Sharpe ratio is *not* leverage invariant.

In this article, we highlight two features of a levered strategy that contribute to the difference between its Sharpe ratio and the Sharpe ratio of its source portfolio. The first is transaction costs. Both leverage-induced trading costs and financing costs diminish Sharpe ratio; see Anderson et al. (2012, Formula 1). The second is the covariance term. Since the covariance term was negative in the examples considered in this article, it lowered the Sharpe ratios of the dynamically levered strategies relative to the Sharpe ratios of their source portfolios and comparably calibrated fixed levered strategies. However, as indicated in Figure 2, the correlation between leverage and the return to the source portfolio, which is the driver of the covariance term, can be highly unstable at horizons of three to five years. So unless a leverage-seeking investor has a specific reason to believe this correlation will be positive over a particular period for a particular dynamically levered strategy, or unless he or she enjoys the coin-flip-like risk illustrated in Figure 2, that investor may prefer a fixed leverage strategy.

Table 8: Historical Performance

Sample Period: 1929-2012 Source: Various, Target: Various r^b = 3M-EDR	Arithmetic Total Return	Geometric Total Return	Average Leverage	Volatility	Arithmetic Excess Return	Sharpe Ratio	Skewness	Excess Kurtosis
60/40	8.18	7.77	1.00	11.58	4.69	0.40	0.19	7.44
VW Market	8.12	7.24	1.00	14.93	4.63	0.31	0.61	14.39
Stocks	10.43	9.00	1.00	18.93	6.95	0.37	0.18	7.46
Risk Parity	5.68	5.74	1.00	4.20	2.20	0.52	0.05	4.92
Bonds	5.08	5.14	1.00	3.26	1.59	0.49	0.03	4.74
$\text{UVT}_{60/40}$	6.85	6.37	3.66	11.54	3.37	0.29	-0.43	2.23
$\text{FLT}_{60/40,\lambda}$	9.19	8.29	3.69	15.53	5.70	0.37	-0.01	4.78
$\text{FLT}_{60/40,\sigma}$	8.03	7.62	2.75	11.57	4.54	0.39	0.00	4.80
$\text{CVT}_{60/40}$	7.56	7.03	3.31	12.22	4.07	0.33	-0.41	7.13
UVT_{MKT}	7.45	6.53	4.71	14.88	3.97	0.27	-0.44	2.23
CVT_{MKT}	7.68	6.52	3.58	16.13	4.19	0.26	-0.75	15.62
$\text{UVTB}_{\text{STOCKS}}$	3.55	1.70	7.43	19.10	0.07	0.00	-0.55	4.75
$\text{FLTB}_{\text{STOCKS},\lambda}$	8.80	5.93	7.49	24.47	5.31	0.22	-0.08	4.68
$\text{FLTB}_{\text{STOCKS},\sigma}$	8.10	6.49	5.80	18.95	4.61	0.24	-0.07	4.66

Table 8: Annualized arithmetic and geometric returns, volatility and Sharpe ratio, of the source portfolios (unlevered risk parity and U.S. Treasury bonds, volatility targets (fully invested 60/40, value-weighted market, and stocks) and the various levered strategies considered in this paper, over the period 1929—2012. Arithmetic returns were estimated from monthly data and annualized by multiplication by 12; they are displayed in percent. Geometric returns are also displayed in percent and were annualized by $(1 + G[r])^{12} - 1$. Volatility was measured from monthly returns and annualized by multiplying by $\sqrt{12}$. Sharpe ratios were calculated using annualized excess arithmetic return and annualized volatility.

6 The Covariance Term, Revisited

The most novel part of our analysis is its focus on the covariance between leverage and excess borrowing return. In this section, we examine the covariance term from the standpoint of volatility targeting. We have already noted that leverage reduces the Sharpe ratio if the borrowing rate exceeds the risk-free rate, or if trading incurs costs. However, in a multi-period setting, leverage has an impact on Sharpe ratio even in the absence of those market frictions, via the covariance term. In order to focus on the covariance term, we make the highly unrealistic assumptions that borrowing is at the risk-free rate (i.e. $r^b = r^f$), which is fixed, and that trading costs are zero. We find that applying UVT leverage *does* change the Sharpe ratio, even under these assumptions.[27] Variable leverage, as used in UVT, is "an unintended market-timing strategy."[28]

Under these unrealistic assumptions, the excess return of the levered strategy is given by:

$$r^{\mathbf{L}} - r^f = \lambda \left(r^{\mathbf{S}} - r^f \right) \tag{9}$$

Suppose we pick a fixed volatility target V; then we must set $\lambda = \dfrac{V}{\text{volatility of source}}$. Thus, we have:

Sharpe ratio of levered strategy

$$
\begin{aligned}
&= \frac{E\left[r^{\mathbf{L}} - r^f\right]}{V} \\
&= \frac{E\left[\lambda \left(r^{\mathbf{S}} - r^f\right)\right]}{V} \\
&= \frac{E\left[\frac{V}{\text{volatility of source}} \times \left(r^{\mathbf{S}} - r^f\right)\right]}{V} \\
&= \frac{E\left[V \times \frac{r^{\mathbf{S}} - r^f}{\text{volatility of source}}\right]}{V} \\
&= E\left[\frac{r^{\mathbf{S}} - r^f}{\text{volatility of source}}\right] \\
&= E\left[r^{\mathbf{S}} - r^f\right] E\left[\frac{1}{\text{volatility of source}}\right] + \mathrm{cov}\left(r^{\mathbf{S}} - r^f, \frac{1}{\text{volatility of source}}\right) \quad (10)
\end{aligned}
$$

Formula (10) makes it clear that a covariance term will be present in the Sharpe ratio of

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[27]This issue has been misunderstood in the published literature. For example, Asness et al. (2013) wrote, "Scaling the returns to any stable risk target (or not scaling them at all) cannot mathematically affect the Sharpe ratio, or the t-statistic of the alpha of our levered portfolios, because we are multiplying the return stream by a fixed constant." Their analysis conflated single-period models with multi-period models, and misstated the construction of the UVT_{MKT} strategy used in Asness et al. (2012).

[28]Asness et al. (2013) asserted that variable *volatility*, rather than variable leverage, is "an unintended market-timing strategy."

any strategy that involves levering a source portfolio of variable volatility to a volatility target. Our empirical examples show that the covariance in Formula (5) has a material effect on realized return and realized Sharpe ratio. It follows that the covariance in Formula (10) has a material effect on realized return and realized Sharpe ratio. Recall that the leverage was especially volatile in the levered strategy UVTB$_{\text{STOCKS}}$, which levered a source portfolio of U.S. Treasury bonds to the volatility of stocks; even though the target volatility was constant, the leverage was very volatile precisely because the inverse of the volatility of bonds was high.

7   Conclusion

In this article, we developed a platform that supports both backward-looking performance attribution and forward-looking investment decisions concerning levered strategies. Specifically, in Formula (6), we expressed the difference between arithmetic expected return to a levered strategy portfolio and its source portfolio as a sum of four terms:

$$E\left[r^{\mathbf{L}}\right] = E\left[r^{\mathbf{S}}\right] + E\left[\lambda - 1\right] E\left[r^{\mathbf{S}} - r^{b}\right] + \text{cov}(\lambda, r^{\mathbf{S}} - r^{b}) - \left(E\left[r^{\mathbf{TCS}}\right] + E\left[r^{\mathbf{TCL}}\right]\right).$$

The first two terms, whose sum we have called *magnified source return*, are the ones that most easily come to mind in the context of a levered strategy. However, as we have shown empirically, other factors have a material effect on the cumulative return to a levered strategy. These include the covariance of leverage with the excess borrowing return, trading costs and compounding effects.

Formula (6) accounted for both the covariance term and transaction costs. However, it neglected the effect of compounding, which imposes a variance drag on cumulative return that is not captured in arithmetic expected return. If the levered strategy has high volatility, the variance drag may be substantial. Hence a more accurate decision rule depends on geometric expected return in Formula (8):

$$G[r] \sim (1 + E[r])e^{-\frac{\text{var}(r)}{2}} - 1.$$

We used Formulas (6) and (8) to examine the realized performance of fixed leverage (FLT) strategies and two dynamically levered strategies: unconditional volatility targeting (UVT) and conditional volatility targeting (CVT). Some scholars have expressed the view that CVT strategies are poor alternatives to UVT strategies;[29] this view is not supported by the results reported in Tables 4 and 6. In fact, it is the leverage that was implicitly determined by the volatility targets in UVT$_{60/40}$ and CVT$_{60/40}$, and not the volatility itself, that interacted with the return to the source portfolio to determine strategy performance. In our 1929–2012 period, CVT$_{60/40}$ outperformed UVT$_{60/40}$. Future work is required to determine whether the sign of the covariance term might be predictable at longer horizons.

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[29]See Asness et al. (2013).

In the examples we considered, the cumulative effects of borrowing and trading costs, the variance drag and the covariance term offset much of the benefit of return magnification. Leverage, both fixed and dynamic, substantially lowered Sharpe ratios. In addition, dynamic leverage added noise to returns. Over our 84-year time horizon, fixed leverage strategies outperformed volatility-targeting strategies, and levered strategies had lower Sharpe ratios than their unlevered source portfolios.

Asness et al. (2012) argued that risk parity (levered to the volatility of the market) outperformed 60/40 over a long horizon;[30] our analysis does not support this.[31] Risk parity performed relatively well over the period 2008–2012, which featured Fed-supported interest rates that were extraordinarily low by historical standards. But that need not indicate how risk parity will perform in other regimes. Rising interest rates tend to raise the cost of funding a levered strategy and lower the prices of bonds in risk parity portfolios at the same time. Rising interest rates also have the potential to limit corporate profits and thereby exert downward pressure on equity prices. These considerations should be incorporated into any decision to lever low-risk portfolios when interest rates are unusually low.

Appendices

A Related Literature

A.1 CAPM

Finance continues to draw heavily on the Capital Asset Pricing Model (CAPM) developed in Treynor (1962), Treynor and Black (1976), Sharpe (1964), Lintner (1965a), Lintner (1965b), Mossin (1966), and extended in Black and Litterman (1992).[32] In the CAPM, leverage is a means to adjust the level of risk in an efficient portfolio and nothing more. In contrast, Markowitz (2005) illustrated another facet of leverage in the context of a market composed of three coconut farms. In this disarmingly simple example, some investors were leverage-constrained and others were not. The market portfolio was mean-variance inefficient; as a result, no mean-variance investor would choose to hold it, and expected returns of assets did not depend linearly on market betas.

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[30]It is reasonable to compare the performance of the value-weighted market to risk parity levered to the volatility of the market, and to compare the performance of 60/40 to risk parity levered to the volatility of 60/40. However, it does not seem reasonable to compare the performance of 60/40 to that of risk parity levered to the volatility of the market; we are grateful to Patrice Boucher for this insight.

[31]See, however, the discussion in Section 4.

[32]A history of the CAPM elucidating Jack Treynor's role in its development is in French (2003).

A.2 Measurement of Risk and Nonlinearities

An impediment to a clear understanding of leverage may be the way we measure its risk. Standard risk measures such as volatility, value at risk, expected shortfall, and beta scale linearly with leverage. But as we know from the collapse of Long Term Capital, the relationship between risk and leverage can be non-linear; see, for example, Jorion (2000). Föllmer and Schied (2002) and Föllmer and Schied (2011, Chapter 4) described risk measures that penalize leverage in a super-linear way. Recent experience suggests that these measures may be useful in assessing the risk of levered strategies.

One contribution of this paper is to explain how the interaction between leverage and market frictions creates specific nonlinearities in the relationship between leverage and return. Understanding these specific nonlinearities provides a practical framework to guide the decision on whether and how to lever.

A.3 Motivations for Leverage

If investors are overconfident in their predictions of investment returns, they may find leverage attractive because it magnifies the returns when times are good, and because they underestimate the risk of bad outcomes.[33]

Perfectly rational investors may also be attracted to leverage by the low risk anomaly, the apparent tendency of certain low-risk portfolios to have higher risk-adjusted return than high-risk portfolios. An investor who believes in the low risk anomaly will be tempted to lever low-risk portfolios, in the hope of achieving high expected returns at acceptable levels of risk.

In a CAPM world, investors with below-average risk aversion will choose to lever the market portfolio.[34] The low risk anomaly provides a rational argument for investors with typical risk aversion to use leverage. Indeed, the low risk anomaly is arguably the only rational argument for an investor to use leverage in an investment portfolio composed of publicly traded securities.[35] Differences in risk aversion could explain some investors choosing higher expected return at the price of higher volatility, but there is little reason for a rational investor to choose *leverage* unless the source portfolio being levered offers superior *risk-adjusted* returns, at a volatility *below* the investor's risk tolerance.

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[33]A positive relationship between overconfident CEOs and firm leverage is documented in Malmendier et al. (2011). Shefrin and Statman (2011) identified excessive leverage taken by overconfident bankers as a contributor to the global financial crisis.

[34]Note, however, that the market portfolio in CAPM includes bonds and other risky asset classes, rather than just stocks. Levered strategies include the use of margin, and futures and other derivatives, to assemble levered equity-only portfolios, which behave quite differently from levered portfolios in CAPM.

[35]There are, of course, other rational arguments for using leverage in other contexts. The leverage provided by a mortgage may be the only feasible way for a household to buy a house, which provides a stream of consumption benefits and tax advantages in addition to facilitating an investment in the real estate market. Companies leverage their shareholder equity with borrowing to finance operations, for a variety of reasons, including differences in risk aversion, informational asymmetries, and tax implications.

A.4 Levered Low-Risk Strategies

Low-risk investing refers to a diverse collection of investment strategies that emphasize low beta, low idiosyncratic risk, low volatility or downside protection. The collection of low-risk strategies includes broad asset allocations, but it also includes narrower strategies restricted to a single asset class. An early reference to low-risk investing is Markowitz (1952) who commented that a minimum-variance portfolio is mean-variance optimal if all assets returns are uncorrelated and have equal expectations. But low-risk strategies typically require leverage in order to meet expected return targets. In an exploration of this idea, Frazzini and Pedersen (2013) echoed some of the conclusions in Markowitz (2005), and they complemented theory with an empirical study of an implicitly levered equity risk factor that was long low-beta stocks and short high-beta stocks. This factor descended from Black et al. (1972), which provided evidence that the CAPM may not properly reflect market behavior.

A.5 Empirical Evidence on Levered Low-Risk Investing

There is a growing empirical literature indicating that market frictions may prevent investors from harvesting the returns promised by a frictionless analysis of levered low-risk strategies. Anderson et al. (2012) showed that financing and trading costs can negate the abnormal profits earned by a levered risk parity strategy in a friction-free market. Li et al. (2014) and Fu (2009) showed that market frictions may impede the ability to scale up the return of low-risk strategies through leverage.[36]

Asset allocation that is based on capital weights has a long and distinguished history; see, for example Graham (1949) and Bogle (2007). However, rules-based strategies that allocate risk instead of, or in addition to, capital are of a more recent vintage. Risk-based investing is discussed in Lörtscher (1990), Kessler and Schwarz (1996), Qian (2005), Clarke et al. (2011), Shah (2011), Sefton et al. (2011), Clarke et al. (2013), Anderson et al. (2012), Cowan and Wilderman (2011), Bailey and de Prado (2012), Goldberg and Mahmoud (2013) and elsewhere. Strategies that target volatility are also gaining acceptance, although the literature is still sparse. Goldsticker (2012) compared volatility targeting strategies to standard allocations such as fixed-mix, and found that the relative performance of the strategies was period dependent.

A.6 The Effect of Leverage on Markets

Another important question is the extent to which leverage may *contribute* to market instability. See, for example, Brunnermeier and Pedersen (2009), Adrian and Shin (2010) and Geanakoplos (2010). We do not address that question here, as we restrict our analysis

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[36]Ross (2004) provided an example of the limits to arbitraging mispricings of interest-only strips of mortgage backed securities.

to the effect of leverage on the return of investment strategies, taking the distribution of the underlying asset returns as given.

A.7 Arithmetic versus Geometric Return

Despite the large literature on the importance of compounding to investment outcomes, analyses of investment strategies are often based on arithmetic expected return. Background references on compounding and geometric return include Fernholz (2002) and MacLean et al. (2011). Perold and Sharpe (1988) discussed how the interplay among volatility, rebalancing and compound return causes a fully-invested fixed-mix or portfolio-insurance strategy to behave differently from a buy-and-hold strategy with the same initial mix. Booth and Fama (1992) worked out the relationship between the compound return to a fixed-mix portfolio and its constituents, and their results were applied to portfolios that include commodities in Willenbrock (2011). Markowitz (2012) compared six different mean-variance approximations to geometric return.

B Data

The results presented in this paper were based on CRSP stock and bond data from January of 1929 through December of 2012. The aggregate stock return is the CRSP value weighted market return (including dividends) from the table *Monthly Stock–Market Indices (NYSE/AMEX/NASDAQ)* – variable name *vwretd*. The aggregate bond return was the face value outstanding (cross-sectionally) weighted average of the unadjusted return for each bond in the *CRSP Monthly Treasury (Master)* table. In this table, the variable name for the unadjusted return is *retnua* and for the face value outstanding is *iout1r*. All bonds in the table were used, provided the values for both *retnua* and *iout1r* were not missing.

The proxy for the risk-free rate was the *USA Government 90-day T-Bills Secondary Market* rate, provided by Global Financial Data (`http://www.globalfinancialdata.com`), covering the period from January of 1929 through December of 2012. The proxy for the cost of financing leverage was the *U.S. 3-Month Euro-Dollar Deposit* rate, downloaded from the Federal Reserve (`http://www.federalreserve.gov/releases/h15/data.htm`). The 3-Month Euro-Dollar Deposit data is available from January of 1971 through December of 2012. Prior to January of 1971, a constant of 60 basis points was added to the 90-day T-Bill rate.[37] Trading costs were calculated using the procedure described in Appendix C. We assumed the cost of trading was 100 basis points from 1926 to 1955, 50 basis points from 1956 to 1970, and 10 basis points from 1971 onward.

The construction of the unlevered and levered risk parity strategies was exactly as

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[37]The average difference between the 90-day T-Bill Rate and the 3-Month Euro-Dollar Deposit Rate from 1971 through 2012 was 102 basis points. So our estimate of 60 basis points was relatively conservative.

detailed in Anderson et al. (2012). The construction of the bonds levered to stock strategies was the analogue for the case of a single asset class.

Anderson et al. (2012), following Asness et al. (2012), used the volatility of the market as the target for risk parity. Here, we used the volatility of 60/40 as the target, because it provides a more appropriate comparison to traditional strategies used by institutional investors. The return of UVT strategies is particularly sensitive to the volatility target.

C Trading Costs

We estimate the drag on return that stems from the turnover-induced trading required to maintain leverage targets in a strategy that levers a source portfolio S.

At time t, the strategy calls for an investment with a leverage ratio of λ_t. We make the harmless assumption that the value of the levered strategy at t, denoted L_t, is \$1.[38] Then the holdings in the source portfolio, or assets, are $A_t = \lambda_t$. The debt at time t is given by $D_t = \lambda_t - 1$.

We need to find holdings A_{t+1} in the portfolio at time $t+1$ that are consistent with the leverage target λ_{t+1}. This turns out to be a fixed point problem since the trading costs must come out of the investor's equity. Between times t and $t+1$, the value of the source portfolio changes from S_t to S_{t+1} and the strategy calls for rebalancing to achieve leverage λ_{t+1}. Just prior to rebalancing, the value of the investment is

$$A'_t = \lambda_t(1 + r_t^{\mathbf{S}}), \tag{11}$$

the liability has grown to $D'_t = (\lambda_t - 1)(1 + r_t^b)$ and the investor's equity is:

$$\begin{aligned} L'_t &= A'_t - D'_t \\ &= \lambda_t(1 + r_t^{\mathbf{S}}) - (\lambda_t - 1)(1 + r_t^b). \end{aligned} \tag{12}$$

Note that in Formulas (11) and (12), we use the source return $r_t^{\mathbf{S}}$ *gross* of trading costs in the source portfolio.

Let $w_t = (w_{t1}, \ldots, w_{tn})^\top$ denote the vector of relative weights assigned to the n asset classes in the source portfolio at time t, so that $\sum_{i=1}^n w_{ti} = 1$ for all t. Just prior to rebalancing, the weights have changed to $w'_t = (w'_{t1}, \ldots, w'_{tn})^\top$, where $w'_{ti} = \frac{w_{ti}(1 + r_i^t)}{1 + r_t^{\mathbf{S}}}$. At time $t+1$, the strategy is rebalanced according to its rules, which produces holdings of $A_{t+1} w_{t+1}$ in the n asset classes. We let $x_t = (x_{t1}, \ldots, x_{tn})^\top$ denote the vector of dollar amounts of the changes in value due to rebalancing, so that:

$$x_t = A_{t+1} w_{t+1} - A'_t w'_t. \tag{13}$$

---

[38]This assumption is harmless in a linear model of trading costs, which we develop here. It would be inappropriate for a realistic model of market impact.

If we assume a linear model, the cost of trading x_t is $\kappa \|x_t\|_1 = \sum_{i=1}^{n} |x_{ti}|$ for some $\kappa \geq 0$. The cost reduces the investor's equity to:

$$L_{t+1} = L'_t - \kappa \|x_t\|_1$$
$$= \lambda_t(1 + r_t^{\mathbf{S}}) - (\lambda_t - 1)(1 + r_t^b) - \kappa \|x_t\|_1. \tag{14}$$

Now let

$$
\begin{aligned}
g(\alpha) &= \frac{\alpha}{L_{t+1}} - \lambda_{t+1} \\
&= \frac{\alpha}{L'_t - h(\alpha)} - \lambda_{t+1},
\end{aligned}
$$

where $g(\alpha)$ denotes the leverage implied by holding αw_{t+1} in the n assets, taking into account the effect of trading costs on equity L_{t+1}, minus the desired leverage. Assuming that g is defined on the whole interval $[0, \lambda_{t+1}L'_t]$, it is continuous, $g(0) = -\lambda_{t+1} < 0$, and $g(\lambda_{t+1}Lt')$, so by the Intermediate Value Theorem, there exists α_{t+1} such that $g(\alpha_{t+1}) = 0.$[39] The value of α_{t+1} can readily be found by a bisection algorithm, which worked well in all of the empirical situations studied in this paper.[40]

We set $A_{t+1} = \alpha_{t+1}$, so the holdings of the n assets are given by $A_{t+1}w_{t+1} = \alpha_{t+1}w_{t+1}$. The reduction in return due to trading costs is given by:

$$r^{\mathbf{TC}} = \kappa \|\alpha_{t+1}w_{t+1} - A'_t w'_t\|_1. \tag{15}$$

We compute the trading cost incurred by the source portfolio, $E[r^{\mathbf{TCS}}]$ in the same way and define the trading cost due to leverage by

$$E[r^{\mathbf{TCL}}] = E[r^{\mathbf{TC}}] - E[r^{\mathbf{TCS}}]. \tag{16}$$

D Geometric Return

In order to analyze the effects of compounding, Booth and Fama (1992) expressed continuously compounded return in terms of arithmetic return. We have chosen to analyze the effects of compounding using the geometric average of monthly returns. Our Formula (19) for the geometric average of monthly returns is somewhat simpler than the formula for continuously compounded return in Booth and Fama (1992). Both derivations rely on the second-order Taylor expansion approximation of the logarithm.

Let L_t denote the equity in a strategy at month t, where $t = 0, 1, \ldots, T$.

---

[39]Typically, α_{t+1} is uniquely determined; if not, choose the largest value satisfying the equation.

[40]If there is *no* α_{t+1} such that $g(\alpha) = 0$, it means the equity of the strategy is so low that the transaction costs in getting to the desired leverage wipe out the equity. We do not observe such severe drawdown in our empirical examples, but clearly it would be possible with extreme leverage or a very volatile source portfolio.

The correct ranking of realized strategy performance, taking compounding into account, is given by $G[r]$, the geometric average of the monthly returns, minus one:

$$
\begin{aligned}
G[r] &= \left(\frac{L_T}{L_0}\right)^{1/T} - 1 \\
&= \left[\prod_{t=0}^{T-1} \frac{L_{t+1}}{L_t}\right]^{1/T} - 1 \\
&= \left[\prod_{t=0}^{T-1} (1 + r_t)\right]^{1/T} - 1
\end{aligned}
\tag{17}
$$

Because the logarithm is strictly increasing, $\log(1 + G[r])$ induces exactly the same ranking of realized strategy returns as $G[r]$. It is a different ranking than the one induced by $E[r]$ and $\log(1 + E[r])$, requiring a correction term involving $\text{var}(r)$:

$$
\begin{aligned}
\log(1 + G[r]) &= \frac{1}{T} \sum_{t=0}^{T-1} \log(1 + r_t) \\
&\sim \frac{1}{T} \sum_{t=0}^{T-1} \left(r_t - \frac{(r_t)^2}{2}\right) \\
&= \frac{1}{T} \sum_{t=0}^{T-1} r_t - \frac{1}{T} \sum_{t=0}^{T-1} \frac{(r_t)^2}{2} \\
&= E[r] - \frac{\text{var}(r) + (E(r))^2}{2} \\
&\sim \log(1 + E[r]) - \frac{\text{var}(r)}{2}
\end{aligned}
\tag{18}
$$
$$
\tag{19}
$$

$$
G[r] \sim (1 + E[r]) e^{-\frac{\text{var}(r)}{2}} - 1
\tag{20}
$$

Formulas (18) and (19) approximate the logarithm by its quadratic Taylor polynomial. When $r_t > 0$, the Taylor series for logarithm is alternating and decreasing in absolute value for $|r_t| < 1$, so the error in the approximation of $\log(1 + r_t)$ in Formula (18) is negative and bounded above in magnitude by $|r^t|^3/3$ for each month t. When $r_t < 0$, the error is positive and may be somewhat larger than $|r^t|^3/3$. Since the monthly returns are both positive and negative, the errors in months with negative returns will substantially offset the errors in months with positive returns, so the errors will tend not to accumulate over time. The approximation error in annual geometric return was at most one basis point in our risk parity examples (see Table 4) and five basis points in our levered bond examples (see Table 7).

E Words and Formulas

Table 9 presents the formulas accompanying the words in our Performance Attribution Tables 2, 4, 6 and 7.

Table 9: Performance Attribution

Description:	Formula:
Total Source Return (gross of trading costs)	$\mathbb{E}[\mathbf{r^S}]$ (gross of trading costs)
Leverage	$\mathbb{E}[\lambda - 1]$
Excess Borrowing Return	$\mathbb{E}[r^S - r^b]$
Levered Excess Borrowing Return	$\mathbb{E}[\lambda - 1] \cdot \mathbb{E}[r^S - r^b]$
Magnified Source Return	$\mathbb{E}[\mathbf{r^S}] + \mathbb{E}[\mathbf{\lambda - 1}] \cdot \mathbb{E}[\mathbf{r^S - r^b}]$
Volatility of Leverage	$\sigma(\lambda)$
Volatility of Excess Borrowing Return	$\sigma(r^S - r^b)$
Correlation(Leverage,Excess Borrowing Return)	$\rho(\lambda, r^S - r^b)$
Covariance(Leverage,Excess Borrowing Return)	$\mathbf{C}ov(\lambda, r^S - r^b)$
Source Trading Costs	$-\mathbb{E}[r^{TCS}]$
Leverage-Induced Trading Costs	$-\mathbb{E}[r^{TCL}]$
Total Levered Return (arithmetic)	$\mathbb{E}[\mathbf{r^L}]$
Compounded Arithmetic Return (gross)	$(1 + \mathbb{E}[r^L]/1200)^{12}$
Variance Correction	$\exp(-\sigma_{rL}^2/2)$
Variance Drag	$[(1 + \mathbb{E}[r^L]/1200)^{12} \cdot \exp(-\sigma_{rL}^2/2) - 1] \cdot 100 - \mathbb{E}[r^L]$
Approximation Error	$\mathbb{G}[\mathbf{r^L}] - [(1 + \mathbb{E}[r^L]/1200)^{12} \cdot \exp(-\sigma_{rL}^2/2) - 1] \cdot 100$
Total Levered Return (geometric)	$\mathbb{G}[\mathbf{r^L}]$

Table 9: Formulas corresponding to the words used in Performance Attribution Tables 2, 4, 6 and 7.

References

Tobias Adrian and Hyun Song Shin. Liquidity and Leverage. *J. Finan. Intermediation*, 19:418–437, 2010.

Robert M. Anderson, Stephen W. Bianchi and Lisa R. Goldberg. Will My Risk Parity Strategy Outperform? *Financial Analysts Journal*, 68(6):75–93, 2012.

Clifford Asness, Andrea Frazzini and Lasse Heje Pedersen. Leverage Aversion and Risk Parity. *Financial Analysts Journal*, 68(1):47–59, 2012.

Clifford Asness, Andrea Frazzini and Lasse Heje Pedersen. Will My Risk Parity Strategy Outperform?: A Comment. *Financial Analysts Journal*, 69(2):12–15, 2013.

David H. Bailey and Marcos López de Prado. Balanced Baskets: A New Approach to Trading and Hedging Risks. *Journal of Investment Strategies*, 1(4):21–62, 2012.

Fischer Black, Michael C. Jensen and Myron Scholes. The Capital Asset Pricing Model: Some Empirical Tests. In Michael C. Jensen, editor, *Studies in the Theory of Capital Markets*, pages 79–121. Praeger Publishers Inc., 1972.

Fisher Black and Robert Litterman. Global Portfolio Optimization. *Financial Analysts Journal*, September:28–43, 1992.

John C. Bogle. *The Little Book of Common Sense Investing: The Only Way to Guarantee Your Fair Share of Stock Market Returns (Little Books. Big Profits)*. Wiley, 2007.

David G. Booth and Eugene F. Fama. Diversification Returns. *Financial Analysts Journal*, 48(3):26–32, 1992.

Markus K. Brunnermeier and Lasse Heje Pedersen. Market Liquidity and Funding Liquidity. *Review of Financial Studies*, 22(6):2201–2238, 2009.

Roger Clarke, Harindra De Silva and Stephen Thorley. Minimum Variance Portfolio Composition. *Journal of Portfolio Management*, 37(2):31–45, 2011.

Roger Clarke, Harindra De Silva and Stephen Thorley. Minimum Variance, Maximum Diversification and Risk Parity: An Analytic Perspective. *Journal of Portfolio Management*, 39(3):39–53, 2013.

David Cowan and Sam Wilderman. Rethinking Risk: What the Beta Puzzle Tells us About Investing, 2011. GMO White Paper.

E. Robert Fernholz. *Stochastic Portfolio Theory*. Springer, New York, 2002.

Hans Föllmer and Alexander Schied. Convex Risk Measures and Trading Constraints. *Finance and Stochastics*, 6:429–427, 2002.

Hans Föllmer and Alexander Schied. *Stochastic Finance: An Introduction in Discrete Time*. Walter de Gruyter, third edition, 2011.

Andrea Frazzini and Lasse Heje Pedersen. Betting Against Beta. forthcoming in *Journal of Financial Economics*, 2013.

Craig French. The Treynor Capital Asset Pricing Model. *Journal of Investment Management*, 1(2):60–72, 2003.

Fangjian Fu. Idiosyncratic Risk and the Cross-Section of Expected Stock Returns. *Journal of Financial Economics*, 91:24–37, 2009.

John Geanakoplos. The Leverage Cycle, 2010. Cowles Foundation Discussion Paper No.. 1715R.

Lisa R. Goldberg and Ola Mahmoud. Risk Without Return. *Journal of Investment Strategies*, 2(2):111–120, 2013.

Ralph Goldsticker. Volatility Targeting May Miss the Mark, 2012. BNY Mellon Asset Management.

Benjamin Graham. *The Intelligent Investor*. Collins, 1949.

Philippe Jorion. Risk Management Lessons From Long-Term Capital Management. *European Financial Management*, 2000.

Christoph Kessler and Günter Schwarz. Investment management for private investors at SBC - The benchmark, 1996. Swiss Bank Corporation / The Global 2nd Quarter.

Rob Kozlowski. Kansas City Employees pension fund commits to BlackRock for Risk Parity. *Pensions & Investments,* June 20, 2013.

Xi Li, Rodney Sullivan and Luis Garcia-Feijoo. The Limits to Arbitrage and the Low Risk Anomaly. *Financial Analysts Journal*, 70(1):52–63, 2014.

John Lintner. Securities, Prices, Risk and Maximal Gains from Diversification. *The Journal of Finance*, 20(4):13–37, 1965a.

John Lintner. The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets. *Review of Economics and Statistics*, 47(1):587–615, 1965b.

Rudolf Lörtscher. Investment Policy at SBC - basic premises and model, 1990. Swiss Bank Corporation / The Global 10/90.

Leonard C. MacLean, Edward O. Thorp and William T. Ziemba. *The Kelly Capital Growth Investment Criterion*. World Scientific, 2011.

Ulrike Malmendier, Geoffrey Tate and Jon Yan. Overconfidence and Early-Life Experiences: The Effect of Managerial Traits on Corporate Financial Policies. *Journal of Finance*, 66(5):1687–1783, 2011.

Harry Markowitz. Portfolio Selection. *Journal of Finance*, VII(1):77–91, 1952.

Harry Markowitz. Mean-Variance Approximations to the Geometric Mean. *Annals of Financial Economics*, 7(1), 2012.

Harry M. Markowitz. Market Efficiency: A Theoretical Distinction and So What? *Financial Analysts Journal*, September/October:17–30, 2005.

Jan Mossin. Equilibrium in a Capital Asset Market. *Econometrica*, 34(4):768–783, 1966.

Lorenzo Naranjo. Implied Interest Rates in a Market with Frictions, 2009. Working paper, New York University.

Andre F. Perold and William F. Sharpe. Dynamic Strategies for Asset Allocation. *Financial Analysts Journal*, 44(1):16–27, 1988.

Edward Qian. Risk Parity Portfolios: Efficient Portfolios Through True Diversification, 2005. PanAgora.

Stephen Ross. *Neoclassical Finance*. Princeton University Press, 2004.

James Sefton, David Jessop, Giuliano De Rossi, Claire Jones and Heran Zhang. Low-Risk Investing, 2011. UBS Investment Research.

Ronnie R. Shah. Understanding Low Volatility Strategies: Minimum Variance, 2011. Dimensional Fund Advisors.

William Sharpe. Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk. *Journal of Finance*, 19(3):425–442, 1964.

Hersh Shefrin and Meir Statman. Behavioral Finance in the Financial Crisis: Market Efficiency, Minsky, and Keynes, 2011. Santa Clara University.

Rodney Sullivan. Speculative Leverage: A False Cure for Pension Woes. *Financial Analysts Journal*, 66(3):6–8, 2010.

Jack Treynor. Toward a Theory of Market Value of Risky Assets, 1962. Presented to the MIT Finance Faculty Seminar in 1962, finally published in 1999 in *Asset Pricing and Portfolio Performance*, Robert J. Korajczyk (editor) London: RiskBooks, 15–22.

Jack L. Treynor and Fischer Black. Capital Market Theory and the Pricing of Financial Securities. In Stewart C. Myers, editor, *Modern Developments in Financial Management*, pages 310–327. Praeger Publishers, 1976.

Scott Willenbrock. Diversification Return, Portfolio Rebalancing, and the Commodity Return Puzzle. *Financial Analysts Journal*, 67(4):42–49, 2011.