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Time-varying risk premia[☆]

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ABSTRACT

Time-varying risk premia (TVRP) is one of the four sources of stock return autocorrelation. TVRP arises in a securities market equilibrium when the equilibrium expected returns of the available investments vary over time; in particular, the presence of TVRP does not indicate pricing inefficiency. This paper provides equilibrium upper bounds on TVRP, as a function of the return period, the time horizon over which the autocorrelations are calculated, and the variability of risk premia. These bounds on TVRP, in combination with the methods of Anderson et al. (2010), allow one to establish lower bounds on the contribution of partial price adjustment, and thus pricing inefficiency, to stock return autocorrelation.

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1. Introduction

Andreu Mas-Colell has made many fundamental contributions to mathematical economics, most notably in general equilibrium theory and game theory. One of his most important contributions within general equilibrium theory is to the existence of equilibrium with infinite-dimensional commodity spaces, in Mas-Colell (1986) and other works. Continuous-time financial markets are an important example of infinite-dimensional commodity spaces, and Mas-Colell's work in the more general setting has played an important role in the development of equilibrium theory in the specific setting of continuous-time financial markets.

When continuous-time financial markets are studied from an equilibrium viewpoint, the prices of the securities are determined endogenously by equating supply and demand. In particular, the

expected return and the volatility of the securities are jointly determined as part of the equilibrium. As the market evolves, the wealths of the agents change, which affects their attitudes toward risk. As a result, the risk premium in the expected return of stocks will vary over time.

One of the robust empirical regularities in finance is that stock returns exhibit substantial autocorrelation, which at first sight suggests pricing inefficiency. However, there are sources of stock return autocorrelation other than pricing inefficiency. See Anderson et al. (2010) for a discussion of the many studies documenting autocorrelation, and the large but inconclusive literature attempting to determine whether autocorrelation represents pricing inefficiency.

Anderson et al. (2010) provide methods to decompose stock return autocorrelation into four components: bid-ask bounce (BAB), the nonsynchronous trading effect (NT), partial price adjustment (PPA), and time-varying risk premia (TVRP). Of these four components, only PPA indicates securities pricing inefficiency, so estimating the magnitude of PPA is of considerable interest. Anderson, Eom, Hahn and Park's methods identify a component of return autocorrelation that can only come from PPA and TVRP, and demonstrate that it is an important source, and in some cases the main source, of stock return autocorrelation. However, their method does not directly distinguish between PPA and TVRP. Thus, TVRP may be viewed as inducing a bias in their measurement of the role of PPA in stock return autocorrelation.

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Since TVRP arises endogenously in equilibrium, equilibrium considerations induce bounds on the magnitude of TVRP. In this paper, we translate those bounds on TVRP into bounds on the bias induced by TVRP in the measurement of PPA, as a function of the time horizon, return period, and the variability of risk premia. As the reader will see, the derivation involves stochastic calculus and somewhat delicate error estimates.

In the specific empirical context considered in Anderson, Eom, Hahn and Park (daily return autocorrelations calculated over a two-year period, so the return period is one day and the time horizon is two years), we show that the bias induced by TVRP in the measurement of PPA is negligible. However, in other contexts, the contribution of TVRP to return autocorrelation is potentially large enough to matter, so estimates of the role of PPA in return autocorrelation should be adjusted.

It is important to distinguish between time-varying expected rates of return and TVRP. Under the assumption that stock prices follow one of the standard processes in finance (such as a Geometric Itô or Geometric Lévy Process), rejection of the hypothesis that stock return autocorrelation is zero is equivalent to rejection of the hypothesis that the expected rate of return is constant. In other words, if we impose the assumption that the return in each period is composed of an expected rate of return plus a volatility term, where the volatility term is uncorrelated with the returns in disjoint periods, then returns are uncorrelated if and only if the expected rate of return is constant. As noted by Campbell et al. (1997, p. 66), the “ R^2 of a regression of returns on a constant and its first lag is the square of the slope coefficient, which is simply the first-order autocorrelation.” As a consequence, if the first-order autocorrelation coefficient of return is α , the proportion of the variation in return that “is predictable using the preceding day’s . . . return” is α^2 . Thus, time-varying expected rates of return and return autocorrelation are simply different faces of a single phenomenon.

TVRP is a special case of time-varying expected rates of return. Suppose that stock prices follow Itô Processes of the form¹

$$\frac{dS(t)}{S(t)} = \mu(t)dt + \sigma(t)dW(t) \quad (1)$$

where $W(t)$ is a standard Wiener process. The absence of arbitrage is equivalent to the existence of a vector process $\lambda(t)$ of prices of risk such that

$$\mu(t) - r(t) = \sigma(t)\lambda(t)^T \quad (2)$$

Here, $\mu(t)$ is the vector process of instantaneous expected rates of return and $r(t)$ is the instantaneous risk-free rate. The instantaneous expected rate of return $\mu(t)$ will vary as a result of changes in $r(t)$, $\sigma(t)$, and $\lambda(t)$, and the resulting variation in $\mu(t)$ cannot be exploited by arbitrage; this is the variation attributable to TVRP. Any variation in μ that does not conform to Eq. (2) can be exploited by arbitrage. Equilibrium pricing processes are always arbitrage-free, and provide tighter bounds on TVRP than can be obtained from arbitrage considerations alone.

Assuming the securities prices process is as given in Eq. (1), the theoretical autocorrelation of $\sigma(t)dW(t)$ is zero: we would like to take daily, weekly or monthly samples of $\sigma(t)\Delta W(t)$ and test for autocorrelation. The problem is that neither $\mu(t)$ nor $W(t)$ is observable. What is observable is the realized rate of return $\mu(t)\Delta t + \sigma(t)\Delta W(t)$, and its autocorrelation. If $\mu(t)$ varies over time, we get a biased estimate of the autocorrelation of $\sigma(t)\Delta W(t)$: standard autocorrelation tests will pick up time-varying expected rates

of return and identify it as autocorrelation, whether or not it comes from TVRP.

This paper makes two main contributions. First, it provides a theoretical formula for the bias in terms of the variability of expected returns, whatever the source of the variability. Second, it uses equilibrium considerations to estimate the magnitude of the bias coming from TVRP.

Suppose a stock has a high expected rate of return. If no one knew the expected rate of return was high, there would be nothing pushing the stock higher, and it would stay relatively stable until the good news underlying the high expected rate of return were announced, at which point the stock price would rise abruptly. If it were widely known that the expected rate of return was high, then many traders would buy the stock, forcing the price to rise abruptly until the future expected rate of return was reduced to the appropriate risk-adjusted level. These abrupt rises in price would be captured econometrically as volatility, and not as autocorrelation. In equilibrium, if we see a string of positive returns establishing statistically significant positive autocorrelation, after eliminating NT and BAB, it can only come from two sources: a period of high equilibrium risk premia (since equilibrium prices are arbitrage-free, this must be TVRP), or the strategic decision of a small group of informed traders with positive information to exercise their informational advantage slowly (PPA). Similarly, if we see a string of negative returns establishing statistically significant positive autocorrelation, after eliminating NT and BAB, it can only come from TVRP and PPA.

In some cases, PPA may result in negative autocorrelation. For example, uninformed traders may attempt to exploit the information of informed traders using momentum strategies, which may lead to overshooting and statistically significant negative autocorrelation. As we shall see below, the bias in the Pearson correlation coefficient induced by TVRP is distributed roughly symmetrically around a positive mean. At any given level of significance, the bias induced by TVRP decreases the probability that the Pearson correlation will be significant and negative. Thus, if we find statistically significant negative autocorrelation after eliminating NT and BAB, we have statistically significant confirmation of negative PPA.

The bias resulting from TVRP in the measured autocorrelation depends on the return period, the time horizon over which the autocorrelations are calculated, and the variability of the risk premium over the time horizon:

- The bias becomes larger as the time horizon increases, because the variation of risk premium is larger over longer time horizons.
- The bias becomes larger as the return period increases. Daily returns are much noisier than yearly returns, so the bias coming from variation in mean returns represents a smaller fraction of daily volatility than of volatility over long return periods.
- The effect of a given size bias on hypothesis tests increases as the number of return periods per time horizon increases. For example, suppose we look at daily return autocorrelation over a two-year time horizon, so we have roughly $n = 500$ days and $n - 1 = 499$ daily returns. The standard error in the autocorrelation tests decreases as n increases, so a bias of a given size represents a larger multiple of the standard error as n gets larger.

We find that under plausible assumptions on the variability of risk premia, the bias in daily returns over a two-year time horizon is very small; however, the bias could be substantial in other settings, and point estimates and hypothesis tests need to be corrected in those settings.

¹ A similar analysis holds if prices follow other standard processes.

2. The bias induced by TVRP

Theorem 2.1. Suppose that the price $S(t)$ of a security or portfolio follows the stochastic differential equation

$$\frac{dS(t)}{S(t)} = \mu(t) dt + \sigma(t) dW(t)$$

where W is a standard Brownian Motion and μ and σ are continuous deterministic functions of time, measured in years. Suppose further that we calculate the Pearson autocorrelation coefficient r_p with a return period of d trading days (thus, $d=1$ corresponds to daily return autocorrelation, $d=5$ weekly return autocorrelation, and so forth) and a time horizon of y years, where each year is assumed to contain 250 trading days; each time horizon contains $n=250y/d$ days. We let $\mu_k = \mu(kd/250)$, $\sigma_k = \sigma(kd/250)$, $W_k = W(kd/250)$ and $\Delta W_k = W_{k+1} - W_k$ denote the mean, volatility, level of the Brownian Motion and change of the Brownian Motion at the k th return period. Let r_{pc} denote the Pearson autocorrelation coefficient corrected to factor out the effects of TVRP (i.e. calculated solely from the volatility terms):

$$r_{pc} \simeq \frac{A}{B} = \frac{(1/(n-2)) \sum_{k=1}^{n-2} \sigma_k \sigma_{k+1} \Delta W_k \Delta W_{k+1} - \bar{v}^2}{(1/(n-2)) \left((\sigma_1^2 (\Delta W_1)^2 / 2) + \sum_{k=2}^{n-2} (\sigma_k^2 (\Delta W_k)^2 + (\sigma_{n-1}^2 (\Delta W_{n-1})^2 / 2) \right) - \bar{v}^2}$$

where

$$\bar{v} = \frac{1}{n-2} \left(\frac{\sigma_1 \Delta W_1}{2} + \sum_{k=2}^{n-2} \sigma_k \Delta W_k + \frac{\sigma_{n-1} \Delta W_{n-1}}{2} \right)$$

Then r_{pc} is not observable, but is related to the observable r_p by the following equations:

$$r_p \simeq \frac{A+Z}{B+Z} \quad \text{where}$$

$$Z = Z_1 + Z_2 = \frac{1}{y} \int_0^y \left(\frac{(\mu(t) - \bar{\mu})d}{250} \right)^2 dt + \frac{2}{n-2} \int_0^y \left(\frac{(\mu(t) - \bar{\mu})d}{250} \right) \sigma(t) dW(t)$$

$$\bar{\mu} = \frac{1}{n-2} \left(\frac{\mu_1}{2} + \sum_{k=2}^{n-2} \mu_k + \frac{\mu_{n-1}}{2} \right) \simeq \frac{1}{y} \int_0^y \mu(t) dt$$

Remark 2.2. Theorem 2.1 tells us that the observed Pearson correlation coefficient r_p is a biased version of the unobservable corrected coefficient r_{pc} . In the absence of TVRP, Z_1 and Z_2 are identically zero. Notice that neither Z_1 nor Z_2 depends on the rate at which μ changes, only on the distribution of μ and (in the case of Z_2) the correlation between μ and σ . Z_1 is a nonnegative constant, the variance of the mean return per return period (day, week, month etc.). Z_2 is normally distributed with mean zero and standard deviation

$$\sigma_Z = \frac{2}{n-2} \sqrt{\int_0^y \left(\frac{(\mu(t) - \bar{\mu})}{250} \right)^2 \sigma^2(t) dt}$$

Moreover, the conditional distribution of Z_2 , conditional on A and B , is asymptotically normal. Anderson et al. (2010) make two types of tests:

- The first type involve one-sided or two-sided tests of portfolio return autocorrelation. In those cases where the hypothesis is rejected, the rejections are usually overwhelming and it is easy to see that the small bias induced by TVRP, i.e. by Z , cannot make any difference in those results.
- The second type compute a large number of individual stock return autocorrelations, and count the number of stocks in which the hypothesis is rejected at the one-sided 2.5% level. The effect of the bias on these tests is more delicate, and we need to carefully estimate the effect of the bias on the expected number of rejections.
 - $|Z_2|$ is typically larger than Z_1 . However, we shall see that Z_2 induces a smaller bias in the expected number of rejections because Z_2 can be either positive or negative. The presence of Z_2 leads the actual test using r_p to reject in some cases in which the correct test using r_{pc} does not reject, and vice versa. The symmetry of Z_2 will imply that the two effects very nearly cancel.
 - If the null hypothesis is that the autocorrelation (corrected for TVRP) is less than or equal to zero, then since $Z_1 \geq 0$, and $|A| \leq B$, the test involving r_p rejects more often than the correct but unobservable test involving r_{pc} , but we can quantify and correct for the bias.
 - If the null hypothesis is that the autocorrelation (corrected for TVRP) is greater than or equal to zero, the test involving r_p rejects less often than the correct but unobservable test involving r_{pc} , so we can ignore the bias.

Proof. The unobservable corrected Pearson sample autocorrelation coefficient is given by

$$r_{pc} = \frac{\sum_{k=1}^{n-2} (\sigma_k \Delta W_k - \bar{v})(\sigma_{k+1} \Delta W_{k+1} - \bar{v})}{\sqrt{\sum_{k=1}^{n-2} (\sigma_k \Delta W_k - \bar{v})^2} \sqrt{\sum_{k=2}^{n-1} (\sigma_k \Delta W_k - \bar{v})^2}}$$

$$\simeq \frac{\sum_{k=1}^{n-2} (\sigma_k \sigma_{k+1} \Delta W_k \Delta W_{k+1}) - 2(n-2)\bar{v}^2 + (n-2)\bar{v}^2}{((\sigma_1 \Delta W_1 - \bar{v})^2 / 2) + \sum_{k=2}^{n-2} (\sigma_{2k} \Delta W_k - \bar{v})^2 + ((\sigma_{n-1} \Delta W_{n-1} - \bar{v})^2 / 2)}$$

$$= \frac{\sum_{k=1}^{n-2} (\sigma_k \sigma_{k+1} \Delta W_k \Delta W_{k+1}) - (n-2)\bar{v}^2}{((\sigma_1 \Delta W_1)^2 / 2) + \sum_{k=2}^{n-2} (\sigma_{2k} \Delta W_k)^2 + ((\sigma_{n-1} \Delta W_{n-1})^2 / 2) - (n-2)\bar{v}^2}$$

$$= \frac{(1/(n-2)) \sum_{k=1}^{n-2} (\sigma_k \sigma_{k+1} \Delta W_k \Delta W_{k+1}) - \bar{v}^2}{(1/(n-2)) \left((\sigma_1 \Delta W_1)^2 / 2 + \sum_{k=2}^{n-2} (\sigma_{2k} \Delta W_k)^2 + (\sigma_{n-1} \Delta W_{n-1})^2 / 2 \right) - \bar{v}^2}$$

$$= \frac{(n-2)A}{(n-2)B} = \frac{A}{B}$$

The observable Pearson sample autocorrelation coefficient is given by

$$r_p = \frac{\sum_{k=1}^{n-2} (((\mu_k d/250) + \sigma_k \Delta W_k) - ((\bar{\mu} d/250) + \bar{v}))((\mu_{k+1} d/250) + \sigma_{k+1} \Delta W_{k+1}) - ((\bar{\mu} d/250) + \bar{v}))}{\sqrt{\sum_{k=1}^{n-2} (((\mu_k d/250) + \sigma_k \Delta W_k) - ((\bar{\mu} d/250) + \bar{v}))^2} \sqrt{\sum_{k=2}^{n-1} (((\mu_k d/250) + \sigma_k \Delta W_k) - ((\bar{\mu} d/250) + \bar{v}))^2}}$$

$$= \frac{(1/(n-2)) \sum_{k=1}^{n-2} (((\mu_k - \bar{\mu}) d/250) + (\sigma_k \Delta W_k - \bar{v}))((\mu_{k+1} - \bar{\mu}) d/250) + (\sigma_{k+1} \Delta W_{k+1} - \bar{v}))}{(1/(n-2)) \sqrt{\sum_{k=1}^{n-2} (((\mu_k - \bar{\mu}) d/250) + (\sigma_k \Delta W_k - \bar{v}))^2} \sqrt{\sum_{k=2}^{n-1} (((\mu_k - \bar{\mu}) d/250) + (\sigma_k \Delta W_k - \bar{v}))^2}}$$

The numerator is given by

$$A + \frac{1}{n-2} \sum_{k=1}^{n-2} \frac{(\mu_k - \bar{\mu}) d}{250} \frac{(\mu_{k+1} - \bar{\mu}) d}{250} + \frac{1}{n-2} \sum_{k=1}^{n-2} \frac{(\mu_k - \bar{\mu}) d}{250} (\sigma_{k+1} \Delta W_{k+1} - \bar{v}) + \frac{1}{n-2} \sum_{k=1}^{n-2} \frac{(\mu_{k+1} - \bar{\mu}) d}{250} (\sigma_k \Delta W_k - \bar{v})$$

$$\simeq A + \frac{1}{y} \int_0^y \left(\frac{(\mu(t) - \bar{\mu}) d}{250} \right)^2 dt + \frac{2}{n-2} \int_0^y \left(\frac{(\mu(t) - \bar{\mu}) d}{250} \right) \sigma(t) dW(t)$$

$$= A + Z$$

The denominator is given by

$$\frac{1}{n-2} \sqrt{\sum_{k=1}^{n-2} \left(\frac{(\mu_k - \bar{\mu}) d}{250} + (\sigma_k \Delta W_k - \bar{v}) \right)^2} \sqrt{\sum_{k=2}^{n-1} \left(\frac{(\mu_k - \bar{\mu}) d}{250} + (\sigma_k \Delta W_k - \bar{v}) \right)^2}$$

$$\simeq B + \frac{1}{n-2} \sum_{k=1}^{n-2} \frac{(\mu_k - \bar{\mu}) d}{250} \frac{(\mu_{k+1} - \bar{\mu}) d}{250}$$

$$+ \frac{1}{n-2} \sum_{k=1}^{n-2} \frac{(\mu_k - \bar{\mu}) d}{250} (\sigma_{k+1} \Delta W_{k+1} - \bar{v}) + \frac{1}{n-2} \sum_{k=1}^{n-2} \frac{(\mu_{k+1} - \bar{\mu}) d}{250} (\sigma_k \Delta W_k - \bar{v})$$

$$\simeq B + Z \quad \square$$

3. Bounding the bias

In this section, we develop specific bounds on the bias. We assume that the number of return subperiods is $n = 250y/d \geq 104$; this is satisfied for monthly returns over periods of nine years or long, weekly returns over periods of two years or longer, and daily returns over periods of five months or longer. When this assumption is not satisfied, the standard errors in the autocorrelation tests are large and it is hard to find significance, whether or not the bias is taken into account; thus, it is not a significant restriction in practice when computing the bound on the bias.

The Pearson test compares $\sqrt{n-2}r_p/\sqrt{1-r_p^2}$ to the standard normal. We have

$$\sqrt{n-2} \frac{r_p}{\sqrt{1-r_p^2}} = \sqrt{n-2} \frac{((A+Z)/(B+Z))}{\sqrt{1-(((A+Z)^2)/((B+Z)^2))}}$$

$$= \sqrt{n-2} \frac{A+Z}{\sqrt{(B+Z)^2 - (A+Z)^2}}$$

Let

$$g_{AB}(Z) = \sqrt{n-2} \frac{A+Z}{\sqrt{(B+Z)^2 - (A+Z)^2}}$$

and let

$$h_{AB}(Z) = \sqrt{\frac{n-2}{B^2 - A^2}} \left(A + \frac{B}{B+A} Z \right)$$

be the first-order Taylor approximation to g_{AB} at $Z=0$.

Noting that A and B are random variables, let $Y = \sqrt{n-2}A/\sqrt{B^2 - A^2}$ be the random variable $g_{AB}(0) = h_{AB}(0)$. Let N denote the cumulative distribution function of the standard normal. As noted in Remark 2.2, we can ignore the bias in the lower-tail one-sided tests, so we focus on one-sided upper tail tests at the 2.5% level. The probability that Z changes an insignificant value to a significant value, using the critical value $\alpha = 0.025$, is the probability that $Y = g_{AB}(0) < \alpha$ and $g_{AB}(Z) \geq \alpha$. Since g_{AB} is increasing in Z , this probability only depends on the value Z_α for which $g_{AB}(Z_\alpha) = \alpha \simeq 1.96$. Recall that we assumed that $n \geq 104$, so $\sqrt{n-2} > 10$.

$$\frac{g_{AB}(Z_\alpha)}{\sqrt{n-2}} \simeq \frac{1.96}{\sqrt{n-2}} < 0.2$$

It is easy to check numerically that for $0 \leq A \leq 0.2B$ and $0 \leq Z \leq 10B$, g_{AB} is concave in Z . If the presence of Z changes an insignificant

value to significant, then $Z \geq 0$ and $Y \leq \alpha \approx 1.96$; since $n \geq 104$, this implies that $A \leq 0.2B$. We assume the following:

$$|Z_2| \leq 4\sigma_Z, \quad g_{AB}(0) < \alpha, \quad g_{AB}(Z) \geq \alpha \Rightarrow A \geq 0, \quad Z \leq 10B \quad (3)$$

and the presence of Z changes an insignificant value to significant, then $A \geq 0$. Eq. (3) thus implies that $g_{AB}(Z_\alpha) \leq h_{AB}(Z_\alpha)$. We believe that Eq. (3) is satisfied in every reasonable empirical situation. We shall show in Section 4 how to check it in a given situation, and find that is overwhelmingly satisfied in the specific empirical situation of Anderson et al. (2010).

The probability that Z changes an insignificant value to a significant value is bounded above by

$$\frac{1}{\sqrt{2\pi}\sigma_Z} \int_0^\infty e^{-z^2/2\sigma_Z^2} \left(N(\alpha) - N\left(\alpha - \frac{\sqrt{n-2}B}{\sqrt{B^2 - A^2(B+A)}}(Z_1 + z)\right) \right) dz \leq 1 - N(4) + \frac{1}{\sqrt{2\pi}\sigma_Z} \int_0^{4\sigma_Z} e^{-z^2/2\sigma_Z^2} (N(\alpha) - N(\alpha - \gamma_1(Z_1 + z))) dz$$

where γ_1 is the value of $\sqrt{n-2}B/(\sqrt{B^2 - A^2(B+A)})$ corresponding to $Y = \alpha - \gamma_1(Z_1 + 4\sigma_Z)$. Although it is hard to solve for γ_1 exactly, we can estimate it using $0 \leq A \leq 0.2B$ as follows:

$$\begin{aligned} & \frac{\partial}{\partial A} \left(\frac{\sqrt{n-2}B}{\sqrt{B^2 - A^2(B+A)}} \right) \\ &= \sqrt{n-2}B \left(\frac{\partial}{\partial A} ((B^2 - A^2)^{-1/2} (B+A)^{-1}) \right) \\ &= \sqrt{n-2}B(A(B^2 - A^2)^{-3/2}(B+A)^{-1} - (B^2 - A^2)^{-1/2}(B+A)^{-2}) \\ &\geq -\sqrt{n-2}B(B^2 - A^2)^{-1/2}(B+A)^{-2} \\ &\geq -\frac{1.0207\sqrt{n-2}}{B^2} \end{aligned}$$

It follows that

$$\gamma_1 \leq \gamma_0 + \frac{1.0207\sqrt{n-2}}{B^2} (Z_1 + 4\sigma_Z)$$

where γ_0 is the value of $\sqrt{n-2}B/\sqrt{B^2 - A^2(B+A)}$ corresponding to $Y = \alpha$; for $\alpha = 1.96$, we have

$$\begin{aligned} A &= \frac{1.96B}{\sqrt{n-2+1.96^2}} \\ \gamma_0 &= \frac{\sqrt{n-2}B}{\sqrt{B^2 - A^2(B+A)}} \\ &= \frac{\sqrt{n-2}}{B} \frac{1}{\sqrt{1 - (1.96^2/(n-2+1.96^2))} \left(1 + (1.96/\sqrt{n-2+1.96^2})\right)} \\ &\leq \frac{\sqrt{n-2}}{B} \frac{1}{\sqrt{1 - (1.96^2/(n-2+1.96^2))}} \\ &\leq \frac{1.019\sqrt{n-2}}{B} \end{aligned}$$

For the specific empirical setting considered in Anderson et al. (2010), $n = 500$: for $n \geq 500$, we can do slightly better:

$$\begin{aligned} \gamma_1 &\leq \gamma_0 + \frac{0.9950\sqrt{n-2}}{B^2} (Z_1 + 4\sigma_Z) \\ \gamma_0 &\leq \frac{.9231\sqrt{n-2}}{B} \end{aligned}$$

Similarly, the probability that Z changes an insignificant value to a significant value is bounded below by

$$\frac{1}{\sqrt{2\pi}\sigma_Z} \int_0^{4\sigma_Z} e^{-z^2/2\sigma_Z^2} (N(\alpha + \gamma_2(z - Z_1)) - N(\alpha)) dz$$

where

$$\gamma_2 \geq \gamma_0 - \frac{1.04318\sqrt{n-2}}{B^2} (4\sigma_Z - Z_1)$$

Thus,

$$0 \leq \gamma_1 - \gamma_2 \leq \frac{8.26\sqrt{n-2}\sigma_Z}{B^2}$$

For $n \geq 500$, 1.04318 improves to 1.009 and 8.26 improves to 8.02.

Using the second order Taylor Expansion for the normal cumulative distribution function, the increase in the probability of rejection resulting from Z is bounded above by

$$\begin{aligned} & \frac{1}{\sqrt{2\pi}\sigma_Z} \int_0^{4\sigma_Z} e^{-z^2/2\sigma_Z^2} (2N(\alpha) - N(\alpha - \gamma_1(Z_1 + z)) - N(\alpha + \gamma_2(z - Z_1))) dz + 1 - N(4) \\ &\leq \frac{1}{\sqrt{2\pi}\sigma_Z} \int_0^\infty e^{-z^2/2\sigma_Z^2} (N'(\alpha)(\gamma_1 + \gamma_2)Z_1 + (\gamma_1 - \gamma_2)z) dz \\ &\quad - \frac{1}{\sqrt{2\pi}\sigma_Z} \int_0^\infty e^{-z^2/2\sigma_Z^2} \frac{N''(\alpha)}{2} ((\gamma_1(Z_1 + z))^2 + (\gamma_2(z - Z_1))^2) dz \\ &\quad + \frac{1}{\sqrt{2\pi}\sigma_Z} \int_0^{4\sigma_Z} e^{-z^2/2\sigma_Z^2} \frac{N'''(\xi(z))}{6} (\gamma_1(Z_1 + z))^3 dz + 10^{-4} \end{aligned}$$

for some measurable function $\xi : [0, \infty) \rightarrow \mathbf{R}$

$$\begin{aligned} &\leq N'(\alpha) \left(\frac{\gamma_1 + \gamma_2}{2} Z_1 + \frac{(\gamma_1 - \gamma_2)\sigma_Z}{\sqrt{2\pi}} \right) \\ &\quad - \frac{N''(\alpha)}{2} \left(\frac{\gamma_1^2 + \gamma_2^2}{2} Z_1^2 + \frac{2\sigma_Z}{\sqrt{2\pi}} (\gamma_1^2 - \gamma_2^2) Z_1 + \frac{\gamma_1^2 + \gamma_2^2}{2} \sigma_Z^2 \right) \\ &\quad + \frac{N'''(\xi(z))\gamma_1^3}{6} \left(\frac{Z_1^3}{2} + \frac{3Z_1^2\sigma_Z}{\sqrt{2\pi}} + \frac{3Z_1\sigma_Z^2}{2} + \frac{2\sigma_Z^3}{\sqrt{2\pi}} \right) + 10^{-4} \\ &\leq \frac{e^{-\alpha^2/2}}{\sqrt{2\pi}} \left(\frac{\gamma_1 + \gamma_2}{2} Z_1 + \frac{(\gamma_1 - \gamma_2)\sigma_Z}{\sqrt{2\pi}} \right) \\ &\quad + \frac{\alpha e^{-\alpha^2/2}}{2\sqrt{2\pi}} \left(\frac{\gamma_1^2 + \gamma_2^2}{2} (Z_1^2 + \sigma_Z^2) + \frac{2\sigma_Z}{\sqrt{2\pi}} (\gamma_1^2 - \gamma_2^2) Z_1 \right) \\ &\quad + \frac{e^{-3/2}\gamma_1^3}{3\sqrt{2\pi}} \left(\frac{Z_1^3}{2} + \frac{3Z_1^2\sigma_Z}{\sqrt{2\pi}} + \frac{3Z_1\sigma_Z^2}{2} + \frac{2\sigma_Z^3}{\sqrt{2\pi}} \right) + 10^{-4} \end{aligned}$$

4. An example

In this section, we present an example showing how to apply the estimates just developed in a particular empirical setting. Anderson et al. (2010) considered the specific situation of daily return auto-correlations of NYSE stocks over two-year time horizons from 1993 through 2008, so we have $d=1$, $n=500$. For now, we focus on the seven two-year subperiods 1993–1994 through 2005–2006; the subperiod 2007–2008, coinciding with the financial crisis of 2007–2009, is discussed separately in Section 4.1.

We must first determine an upper bound on the plausible magnitude of the variation of the risk premium of NYSE stocks and portfolios over a two-year time horizon. Stocks and portfolios could conceivably have expected rates of return below the risk-free rate, but only if they were negatively correlated with undiversifiable

risks and thus provided insurance against those risks. The portfolios considered in Anderson et al. (2010) are stratified by firm size, but they are otherwise diversified. The portfolios are positively correlated with two important undiversifiable risks (the market as a whole and aggregate income), so it is implausible that investors would hold the portfolios if they had an expected rate of return below the risk-free rate. To maintain equilibrium, stock prices would have to fall to raise the future expected rate of return sufficiently to induce stockholders to retain their holdings.

Equilibrium implies the absence of arbitrage, which in term implies (as noted above in Eq. (2)) that $\mu(t) - r(t) = \sigma(t)\lambda(t)^\top$; the equilibrium risk premium depends on the volatility of the securities. The conventional wisdom is that the daily volatility of the broad U.S. stock market is around 1%. Anderson et al. (2010) report in Table 10 the standard deviations of daily returns for SPDRs, the principal Exchange-Traded Fund tracking the Standard & Poor's 500 index: 0.629% (93–94), 0.679% (95–96), 1.277% (97–98), 1.297% (99–00), 1.526% (01–02), 0.906% (03–04), 0.639% (05–06), and 1.904% (07–08). Setting aside the subperiod 2007–2008 of the financial crisis, the average of the standard deviations is 0.993%, very close to the conventional wisdom.

Daily volatility of 1% corresponds to yearly volatility of $\sqrt{249} \approx 15.8\%$. If the expected rate of return of the S&P 500 exceeded the risk-free rate by 15% per annum, investors would surely choose to substantially increase their stockholdings: taking the volatility into account, there is roughly an 82% chance that the S&P 500 will outperform the risk-free rate over a one-year horizon, and only about a 2.5% chance that the S&P 500 will underperform the risk-free rate by 15% or more. Thus, equilibrium considerations dictate that the expected rate of return on the broad stock market should range between the risk-free rate r and $r + 15\%$ for two-year subperiods.

For U.S. stocks, it is natural to take the three-month Treasury Bill Rate as the risk-free rate. Anderson et al. (2010) divided a sixteen-year data period (1993–2008) into eight two-year subperiods, and computed autocorrelations over these two-year time horizons. The variations (max–min) in the three-month Treasury Bill rates for their five two-year subperiods are as follows: 3.34% (6.39–3.05%) (93–94), 1.28% (6.40–5.12%) (95–96), 0.86% (5.83–4.97%) (97–98), 1.99% (6.84–4.85%) (99–00), 4.96% (6.27–1.31%) (01–02), 1.48% (2.51–1.03%) (03–04), 2.85% (5.35–2.50%) (05–06), and 4.45% (5.77–1.32%) (07–08). The average of the subperiod variations is 2.65%. Thus, TVRP should induce variation in the expected return of the portfolios of no more than 18% per annum over a two-year period.

In the case of an individual stock, the expected rate of return should reflect the risk premia of the factors underlying its pricing. Some stocks may have low—even negative—risk premia, while others may have large risk premia. However, the correlation of any given stock with the main risk factors should be relatively stable over time periods of a year or two. Thus, TVRP should induce variation in the expected returns of the individual stocks of no more than 18% per annum over a two-year period.

Assuming that μ is distributed uniformly over an interval of length $18\% = .18$ per annum, $Z_1 = 4.320 \times 10^{-8}$. Assuming that μ is uniformly distributed over an interval of length .18 per annum and σ is constant, $\sigma_Z = 1.180 \times 10^{-6}\sigma$.

The returns of individual stocks are more volatile than returns on the overall market, and returns on smaller stocks are more volatile than returns on larger stocks. However, notice that A and B are quadratic in σ (i.e. if we double the function $\sigma(t)$ at all times, then A and B are quadrupled), while Z_1 is independent of σ and Z_2 is linear in σ . Thus, the bias induced in r_p by Z is maximized when σ is minimized. The returns on individual stocks in a portfolio are more volatile than the returns of the portfolio, and the returns of smaller stocks are more volatile than the returns of larger stocks.

We take the volatility of the S&P 500 index as a lower bound on the volatility of the individual stocks in our analysis.

As noted above, the average value of σ_r for the SPDRs, over the seven two-year subperiods of 1993–2006, is .00993. Assuming σ is constant, we obtain the estimate $\sigma = \sqrt{249} \ln(1.00993) \approx .15592$. With probability $2 \times (1 - N(4)) > 1 - 2 \times 10^{-4}$, $|B - (\sigma_r)^2| = |Z| \leq 4.320 \times 10^{-8} + 4 \times 1.18 \times 10^{-6}\sigma = 7.791 \times 10^{-7}$, so $B = (0.986 \pm .008) \times 10^{-4}$; since the bias is maximized when B is minimized, we assume $B = 0.978 \times 10^{-4}$, $\sigma = .155$, and $\sigma_Z = 1.829 \times 10^{-7}$. $\gamma_2 \leq \gamma_0 = .9231\sqrt{498}/B = 2.106 \times 10^5$, $\gamma_1 \leq \gamma_0 + (.9950\sqrt{498}/B^2)(Z_1 + 4\sigma_Z) = 2.124 \times 10^5$, $0 \leq \gamma_1 - \gamma_2 \leq 8.26\sqrt{498}\sigma_Z/B^2 = 3.42 \times 10^3$, $0 \leq \gamma_1^2 - \gamma_2^2 = (\gamma_1 - \gamma_2)(\gamma_1 + \gamma_2) \leq 1.448 \times 10^9$.

We need to check that Eq. (3) is satisfied. Assuming that $Z_2 \leq 4\sigma_Z$,

$$\frac{Z}{B} \leq \frac{Z_1 + 4\sigma_Z}{B} = \frac{7.748 \times 10^{-7}}{9.78 \times 10^{-5}} < 10^{-2} \ll 10$$

$$g_{AB}(Z) > 0, A < 0 \Rightarrow g_{AB}(0) < 0, g_{AB}(Z) \geq \alpha \Rightarrow g'_{OB}(0) \geq \frac{\alpha}{Z_1 + 4\sigma_Z} = 2.530 \times 10^6$$

$$g'_{OB}(0) = \frac{\sqrt{n-2}}{B} = \frac{\sqrt{498}}{9.78 \times 10^{-5}} = 2.282 \times 10^5 \ll 2.530 \times 10^6$$

Thus, Eq. (3) is satisfied.

Then the increase in the probability of rejection resulting from Z is at most

$$\begin{aligned} & \frac{e^{-\alpha^2/2}}{\sqrt{2\pi}} \left(\frac{\gamma_1 + \gamma_2}{2} Z_1 + \frac{(\gamma_1 - \gamma_2)\sigma_Z}{\sqrt{2\pi}} \right) \\ & + \frac{\alpha e^{-\alpha^2/2}}{2\sqrt{2\pi}} \left(\frac{\gamma_1^2 + \gamma_2^2}{2} (Z_1^2 + \sigma_Z^2) + \frac{2\sigma_Z}{\sqrt{2\pi}} (\gamma_1^2 - \gamma_2^2) Z_1 \right) \\ & + \frac{e^{-3/2}\gamma_1^3}{3\sqrt{2\pi}} \left(\frac{Z_1^3}{2} + \frac{3Z_1^2\sigma_Z}{\sqrt{2\pi}} + \frac{3Z_1\sigma_Z^2}{2} + \frac{2\sigma_Z^3}{\sqrt{2\pi}} \right) + 3 \times 10^{-4} \\ & = .0584 (9.138 \times 10^{-3} + 2.497 \times 10^{-4}) \\ & + .0573 (1.580 \times 10^{-3} + 9.128 \times 10^{-6}) \\ & + 2.133 \times 10^{14} (4.031 \times 10^{-23} + 4.217 \times 10^{-22} \\ & + 2.310 \times 10^{-21} + 5.370 \times 10^{-21}) + 3 \times 10^{-4} \\ & = 5.483 \times 10^{-4} + 0.918 \times 10^{-4} + 0.017 \times 10^{-4} + 3 \times 10^{-4} \\ & = 9.410 \times 10^{-4} \end{aligned}$$

Thus, the bias induced by TVRP increases the probability of rejection from .025 by less than .001–.026, so the expected number of rejections in 100 autocorrelations increases by at most .1 from 2.5 to 2.6. The tests of individual stock autocorrelations and the tests using SPDRs in Anderson et al. (2010) are all based on comparing the number of rejections to 2.5. Changing 2.5 to 2.6 to adjust for the bias increases the p -values slightly but makes no qualitative change in those findings.

4.1. The financial crisis of 2007–2009

The two-year subperiod 2007–2008 coincided with the beginning and height of the financial crisis. As noted above, the daily standard deviation on the SPDRs in 2007–2008 was 1.904%, nearly double the conventional wisdom of 1%. Moreover, the volatility rose over the period, peaking in late 2008 and early 2009. The high volatility of that period should have induced a substantial

increase in risk premia,² and therefore increased the *variability* of risk premia over the period.

However, as noted above, A and B are quadratic in σ , while Z_1 is independent of σ and Z_2 is linear in σ . Thus, the direct effect of increased volatility mitigates some of the additional bias induced by greater variation in risk premia.

More to the point, as noted above, the bias in autocorrelation induced by TVRP is distributed roughly symmetrically around a positive mean, and it always *reduces* the probability of finding statistically negative autocorrelation. While the autocorrelation documented by Anderson et al. (2010) is generally positive in the first half (1993–2002) of the 1993–2008 data period, it becomes negative toward the end of the period, and virtually all of the statistical significance in 2007–2008 comes from negative

autocorrelations. TVRP cannot be the source of these negative autocorrelations in 2007–2008, and eliminating it would strengthen the findings of negative autocorrelation resulting from PPA in 2007–2008. Thus, it is not necessary to do a more careful analysis of the potential impact of the high volatility of 2007–2008 on the variability of risk premia.

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² A rise in risk premia should result in a *decline* in stock prices, as prices must fall to induce people to continue to hold the suddenly riskier stocks. The sharp decline in stock prices over the second half of 2008 and first quarter of 2009 is consistent with a rise in risk premia, while the subsequent recovery in stock prices is consistent with higher but stable risk premia and/or falling risk premia. The conventional wisdom is that risk premia declined after the second quarter of 2009, as the financial services sector stabilized, diminishing the prospect of a contraction as severe as that of the 1930s.