Proof of Lemma 2

Lemma 2 in the online Appendix of the paper "Offshoring in a Ricardian World," establishes that $x_M(\alpha)$ is increasing. To prove this lemma, I first introduce some notation. Let

$$\begin{aligned} H(x,\alpha) &= x^2 + Bx/A - C/A^2 \\ J(x,\alpha) &= \left[\left(1+\frac{\alpha b L_1/L_2}{f(\alpha)}\right) - Ax(1-b)\right]\frac{const}{(1+\alpha)A^2} \end{aligned}$$

where

$$\begin{aligned} A &= \left(\frac{f(\alpha)}{1+\alpha}\right)^b,\ const = \delta^\theta \left(f(\alpha)\right)^b \Phi_{-m}/L_1 \\ B &= (1-b)C - \left(1+\frac{\alpha b L_1/L_2}{f(\alpha)}\right) - b - \frac{\alpha}{(1+\alpha)}b \\ C &= (L_2/L_1)\frac{f(\alpha)}{(1+\alpha)} \end{aligned}$$

Simple algebra shows that $G(x,\alpha) = F(x,\alpha) \Leftrightarrow H(x,\alpha) = J(x,\alpha)$, so $x_M(\alpha)$ solves

$$H(x,\alpha) = J(x,\alpha)$$

The proof that $x_M(\alpha)$ is increasing includes three steps:

1) First, I prove that the solution $x_M^0(\alpha)$ of $H(x,\alpha) = 0$ is increasing in alpha. Since $J(x,\alpha)$ is flat in x if $\Phi_{-m} = 0$ (since $const = 0$) then this implies that if $\Phi_{-m} = 0$ then $x_M(\alpha) = x_M^0(\alpha)$ is increasing in α. The rest of the proof extends this to $\Phi^* > 0$.

2) Next, I prove that if $\alpha_2 > \alpha_1$ then $H(x,\alpha_2) < H(x,\alpha_1)$ for any $x \geq x_M^0(\alpha_1)$.

3) Finally, I prove that the solution of $J(x,\alpha_2) = J(x,\alpha_1)$, where α_2 is greater and close to α_1, is less than $x_M^0(\alpha_1)$.

Thus, given that the slope of J w.r.t. x increases (declines in absolute value) as α increases, then the three steps above are sufficient to prove that $x_M(\alpha)$ is increasing in alpha, since the shift of $J(x,\alpha)$ with an increase in alpha amplifies the effect of increasing α on $x_M^0(\alpha)$.

First step: We want to prove that $x_M^0(\alpha)$ is increasing in alpha. This is done by solving explicitly for the highest solution to $H(x,\alpha) = 0$ and then differentiating w.r.t. α and showing that the result is positive. Given the expression for $H(x,\alpha) = 0$ then $x_M^0(\alpha)$ is determined by the positive solution of

$$A^2x^2 + ABx - C = 0,$$

or

$$x_M(\alpha) = \frac{-B + \sqrt{B^2+4C}}{2A}$$

Differentiation yields:

$$\frac{dx_M(\alpha)}{d\alpha} = \frac{A\left(\frac{2BB'+4C'}{2\sqrt{B^2+4C}} - B'\right) - A'\left(\sqrt{B^2+4C} - B\right)}{2A^2}.$$

It is easy to show that this is positive if and only if

$$(A'B - AB')\left(\sqrt{B^2+4C} - B\right) > A'4C - 2C'A$$

Differentiating to get A' and C' and then plugging in and simplifying reveals that

$$A'4C - 2C'A = 2A\frac{1 + L_2/L_1}{(1+\alpha)^2}(1-2b).$$

Hence, we want to show that

$$\left(\frac{A'}{A}B - B'\right)\left(\sqrt{B^2+4C} - B\right) > 2\frac{1 + L_2/L_1}{(1+\alpha)^2}(1-2b)$$

Now,

$$\frac{A'}{A} = \frac{-b\left(\frac{f(\alpha)}{(1+\alpha)}\right)^{b-1}\frac{1+L_1/L_2}{(1+\alpha)^2}}{\left(\frac{f(\alpha)}{(1+\alpha)}\right)^{b}} = -b\frac{1 + L_1/L_2}{f(\alpha)\,(1+\alpha)}$$

and

$$-B' = (1-b)(L_2/L_1)\frac{1 + L_1/L_2}{(1+\alpha)^2} + bL_1/L_2\frac{1}{(f(\alpha))^2} + \frac{b}{(1+\alpha)^2}.$$

Consider $\sqrt{B^2+4C} - B$ as a function of $b \in (0, 1/2)$. We have

$$\begin{aligned}\left(\sqrt{B^2+4C} - B\right)'_b &= \frac{2BB'}{2\sqrt{B^2+4C}} - B' \\ &= B'\left(\frac{B - \sqrt{B^2+4C}}{\sqrt{B^2+4C}}\right) > 0,\end{aligned}$$

as $B - \sqrt{B^2+4C} < 0$ and $B' < 0$. Thus, it is sufficient to show that

$$\left(\frac{A'}{A}B - B'\right)\left(\sqrt{B^2+4C} - B\right)_{b=0} > 2\frac{1 + L_2/L_1}{(1+\alpha)^2}(1-2b),$$

But

$$\begin{aligned}\left(\sqrt{B^2+4C} - B\right)_{b=0} &= \sqrt{\left((L_2/L_1)\frac{f(\alpha)}{(1+\alpha)} - 1\right)^2 + 4(L_2/L_1)\frac{f(\alpha)}{(1+\alpha)}} - \left((L_2/L_1)\frac{f(\alpha)}{(1+\alpha)} - 1\right) \\ &= (L_2/L_1)\frac{f(\alpha)}{(1+\alpha)} + 1 - \left((L_2/L_1)\frac{f(\alpha)}{(1+\alpha)} - 1\right) = 2.\end{aligned}$$

So, we want to prove that

$$\left(\frac{A'}{A}B - B'\right) > \frac{1 + L_2/L_1}{(1+\alpha)^2}(1-2b)$$

Some manipulation reveals that

$$\begin{aligned}\frac{A'}{A}B - B' &= (1-b)^2\frac{1 + L_2/L_1}{(1+\alpha)^2} + bL_1/L_2\frac{1}{(f(\alpha))^2} \\ &+ \frac{b}{(1+\alpha)^2} + b\frac{1+L_1/L_2}{f(\alpha)\,(1+\alpha)}\left(1 + \frac{\alpha b L_1/L_2}{f(\alpha)} + b + \frac{\alpha}{(1+\alpha)}b\right)\end{aligned}$$

But it is trivial to establish that this is positive.

Second step: Consider equation $H(x,\alpha_1) = H(x,\alpha_2)$ for any $\alpha_i : \alpha_2 > \alpha_1$. It is a linear equation so it has a unique solution. Moreover, so

$$\left(\frac{(L_2/L_1)f(\alpha)}{(1+\alpha)A^2}\right)'_\alpha = L_2/L_1\left(\left(\frac{f(\alpha)}{(1+\alpha)}\right)^{1-2b}\right)'_\alpha .$$

Since $\theta > 1$ (an assumption in EK 2002) $b < 1/2$. That is, $1 - 2b > 0$. This means that $\left(\frac{(L_2/L_1)f(\alpha)}{(1+\alpha)A^2}\right)'_\alpha < 0$ or $-\left(\frac{(L_2/L_1)f(\alpha)}{(1+\alpha)A^2}\right)'_\alpha > 0$. That is, the intercept of $H(x,\alpha)$ with vertical axis is always negative and increasing in α. Thus, $0 > H(0,\alpha_2) > H(0,\alpha_1)$. Since H is U-shaped and $x_M^0(\alpha_2) > x_M^0(\alpha_1) > 0$ (see [1]) then $H(x_M^0(\alpha_1),\alpha_2) < H(x_M^0(\alpha_1),\alpha_1) = 0$.[2] By continuity, there must exist $x^* \in (0, x_M^0(\alpha_1))$ such that $H(x^*,\alpha_1) = H(x^*,\alpha_2)$. Since there is a unique solution to this equation, it follows that $H(x,\alpha_2) < H(x,\alpha_1)$ for all $x \geq x_M^0(\alpha_1)$.

Third step: It is obvious if $J(x,\alpha)$ is fixed and does not change with an increase in alpha, then from the previous two steps we can say that $x_M(\alpha)$ is increasing in alpha. However, with an increase in alpha the curve $J(x,\alpha)$ pivots around some point, with the slope becoming higher or less negative. If we prove that the solution to $J(x,\alpha_2) = J(x,\alpha_1)$ with α_2 just higher than α_1 is less than $x_M^0(\alpha_1)$, then we are done with the proof because the change in $J(x,\alpha)$ amplifies the overall effect on $x_M(\alpha)$. We have

$$J(x,\alpha) = D(\alpha) - F(\alpha)x$$

where

$$\begin{aligned}D(\alpha) &= \left(1 + \frac{\alpha b L_1/L_2}{f(\alpha)}\right)\frac{const}{(1+\alpha)A^2} \\ F(\alpha) &= A(1-b)\frac{const}{(1+\alpha)A^2}\end{aligned}$$

[1] The last inequality comes from $x_M(\alpha) = \frac{-B+\sqrt{B^2+4C}}{2A}$ and noting that $-B + \sqrt{B^2+4C} > -B + \sqrt{B^2} = -B + |B| > -B + B = 0$.

[2] To see this, recall that $x_M^0(\alpha)$ is the highest solution to $H(x,\alpha) = 0$ so that $H_x(x_M^0(\alpha),\alpha) > 0$. Thus, it must be the case that $H(x_M^0(\alpha_1),\alpha_2) < 0$, for otherwise the curve $H(x,\alpha_2)$ would have its lower solution to $H(x,\alpha_2) = 0$ for a level of x higher than $x_M^0(\alpha_1)$ and hence given the U-shape form of H it would follow that $H(0,\alpha_2) > 0$, which is a contradiction.

Then,

$$\begin{aligned} J(x,\alpha_2) &= J(x,\alpha_1) \iff \\ x &= \frac{D(\alpha_1)-D(\alpha_2)}{F(\alpha_1)-F(\alpha_2)} \end{aligned}$$

If we take the limit $\alpha_2 \to \alpha_1$, then

$$x = \frac{D'(\alpha)}{F'(\alpha)}$$

Tedious algebra shows that

$$\frac{D'(\alpha)}{F'(\alpha)} = \frac{1}{1-b}\left(\frac{(1+\alpha)}{f(\alpha)}\right)^b \left\{ \left(1+\frac{\alpha b L_1/L_2}{f(\alpha)}\right) - \frac{(1+\alpha)\left\{\frac{bL_1/L_2}{(f(\alpha))^2} + \left(1+\frac{\alpha b L_1/L_2}{f(\alpha)}\right)\frac{b(1+L_1/L_2)}{(1+\alpha)f(\alpha)}\right\}}{(1-b)} \right\}$$

Next, we compare $\frac{D'(\alpha)}{F'(\alpha)}$ with $x_F(\alpha) = \left(\frac{(1+\alpha)}{f(\alpha)}\right)^b < x_M^0(\alpha)$ (this last inequality follows because $x_F(\alpha) < x_M(\alpha)$ for all Φ_{-m} including $\Phi_{-m} = 0$, but $x_M(\alpha; \Phi_{-m} = 0) = x_M^0(\alpha)$). Algebra shows that this is equivalent to

$$\frac{(1+\alpha)\left\{\frac{L_1/L_2}{(f(\alpha))^2} + \left(1+\frac{\alpha b L_1/L_2}{f(\alpha)}\right)\frac{(1+L_1/L_2)}{(1+\alpha)f(\alpha)}\right\}}{(1-b)} > \frac{\alpha L_1/L_2}{f(\alpha)} + 1$$

The left side of the inequality positively depends on b. Thus, to prove the inequality we can take $b = 0$, and then simple algebra reveals that the inequality holds. Thus, we proved that the solution of $J(x,\alpha_2) = J(x,\alpha_1)$ for α_2 higher but close to α_1 is strictly less than $x_F(\alpha_1) < x_M^0(\alpha_1)$. **Q.E.D.**