University of California, Berkeley **Economics 230b**
Department of Economics **Spring 2001**

Handout #1: Derivation of the User Cost of Capital

Consider a firm wishing to maximize its value, defined as:

(1) $$V_t = \int_{s=t}^{\infty} e^{-r(s-t)} x_s \, ds$$

where x_s is the firm's cash flow from real activities at date s,

(2) $$x_s = (1-t_s) p_s F(K_s) - q_s I_s (1-k_s) + t_s \int_{u=-\infty}^{s} D_u(s-u) q_u I_u \, du$$

In (2), K_s is the capital stock at date s, I_s is the investment flow, p_s is the price of output, and q_s is the price of capital. The tax system has three components: t_s is the corporate tax rate at date s, k_s is the initial subsidy to investment (e.g., an investment tax credit), and $D_u(s-u)$ is the depreciation deduction at date s per dollar of investment at an earlier date u. This deduction depends not only on the age of the asset, $(s-u)$, but also on the tax rules that prevailed at date u.

Inserting (2) into (1) yields:

(3) $$V_t = \int_{s=t}^{\infty} e^{-r(s-t)} \left((1-t_s) p_s F(K_s) - q_s I_s (1-k_s) + t_s \int_{u=-\infty}^{s} D_u(s-u) q_u I_u \, du \right) ds$$

$$= \int_{s=t}^{\infty} e^{-r(s-t)} \left((1-t_s) p_s F(K_s) - q_s I_s (1-k_s) + t_s \int_{u=t}^{s} D_u(s-u) q_u I_u \, du + t_s \int_{u=-\infty}^{t} D_u(s-u) q_u I_u \, du \right) ds$$

$$= \int_{s=t}^{\infty} e^{-r(s-t)} \left((1-t_s) p_s F(K_s) - q_s I_s (1-k_s) + t_s \int_{u=t}^{s} D_u(s-u) q_u I_u \, du \right) ds + \overline{V}_t$$

where the second line breaks the flows of depreciation allowances into two pieces: those attributable to investment after date t and before date t. The second piece affects the value of the firm at date t, but not its decisions from date t onward, and so may be ignored. The remaining expression for firm value can be simplified by changing the order of integration for depreciation allowances (first over date of allowances, then over date of investment, rather than starting with date of investment), leading to:

(4) $$V_t = \int_{s=t}^{\infty} e^{-r(s-t)} \left((1-t_s) p_s F(K_s) - q_s I_s (1-k_s) + q_s I_s \int_{u=s}^{\infty} e^{-r(u-s)} t_u D_s(u-s) \, du \right) ds + \overline{V}_t$$

$$= \int_{s=t}^{\infty} e^{-r(s-t)} \big((1-t_s) p_s F(K_s) - q_s I_s (1-\Gamma_s) \big) ds + \overline{V}_t$$

where $\Gamma_s = k_s + \int_{u=s}^{\infty} e^{-r(u-s)} t_u D_u(u-s) \, du$ is the present value of tax benefits per dollar invested at date s.

Expression (4) provides a general expression for the value of the firm at date *t*. Determining the optimal investment policy requires further specification of the firm's technology. We assume that capital depreciates exponentially, so that:

$$(5) \qquad \dot{K}_t = I_t - dK_t$$

in which *d* is the rate of economic depreciation of capital, and that the full marginal cost of investment, *q*, is not affected by the level of investment. Then, inserting (5) into (4) and solving for an optimum based on the Euler equation, $\frac{\partial V_t}{\partial K_s} - \frac{d\left(\partial V_t / \partial \dot{K}_s\right)}{ds} = 0$, yields:

$$(6) \qquad F'(K_s) = \frac{q_s^*}{p_s} \frac{\left(r + d - \frac{\dot{q}_s^*}{q_s^*}\right)}{(1 - t_s)}; \qquad \text{where } q_s^* = q_s(1 - \Gamma_s)$$

One may think of q^* as the effective price of capital goods, taking into account the present value of tax benefits directly associated with investment.