# EQUILIBRIA IN PRODUCTION ECONOMIES

CHARALAMBOS D. ALIPRANTIS[1], MONIQUE FLORENZANO[2], AND RABEE TOURKY[3]

[1] Department of Economics, Krannert School of Management, Purdue University, 403 West State Street, W. Lafayette, IN 47907-2056, USA;aliprantis@mgmt.purdue.edu

[2] CNRS–CERMSEM, MSE Université Paris 1,106-112 boulevard de l'Hôpital, 75647 Paris Cedex 13, FRANCE; monique.florenzano@univ-paris1.fr

[3] Department of Economics and Commerce, University of Melbourne, Parkville 3052, Melbourne, AUSTRALIA; rtourky@unimelb.edu.au

Abstract. This paper studies production economies having a locally convex topological vector commodity space ordered by a closed and generating convex cone such that the order intervals are topologically bounded. The generally assumed lattice properties on the commodity–price duality are replaced by an assumption of uniform properness of the Riesz–Kantorovich functional associated with a list of continuous linear functionals. On such an economy, our assumptions are quite general. In particular, consumers preferences are non-ordered, not necessarily monotone, and we do not assume free-disposal. The equilibrium existence theorem established in this paper is the most general for production economies in the literature.

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## 1. Introduction

This paper is the last in a series of papers [8, 9, 10] studying Walrasian equilibria without vector lattice assumptions on the commodity–price duality of an economy. More precisely, this paper extends to production economies results obtained in [10] for exchange economies.

In [9] and [10], we were addressing the equilibrium existence problem for exchange economies having ordered topological vector commodity spaces. The main motivation for going from lattice ordered topological vector spaces, the case usually considered in the literature, to this setting was coming from Finance. In security models, one is given with a pair of function spaces, $L$ and $X$, where $L$ is the portfolio space and $X$ is the space of contingent claims, together with a linear

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operator $R\colon L \to X$, one-to-one and positive, which pulls back the order intervals of $R(L)$ to closed and bounded subsets of $L$. The portfolio dominance ordering, a notion introduced in [4, 5], is the only ordering which is relevant for purposes of economic analysis, especially for studying the arbitrage freeness of security prices. But, when the portfolio space is reordered by the portfolio dominance ordering, it is seldom a vector lattice and the existence of equilibrium cannot be deduced from theorems which require lattice properties on the commodity space. Minimal conditions of compatibility between order and topology of the commodity (portfolio) space were given in [9] and [10] for completing a general equilibrium analysis of exchange models.

In this paper, we study the existence of general equilibrium for a production economy having an ordered topological vector commodity space. To go from exchange models to production economies is never easy. It suffices to recall the competitive challenge which lead after the seminal paper of Mas-Colell [27] to the publication in 1987 of Zame's paper[39] and Aliprantis–Brown–Burkinshaw's paper[3] and in 1989 of Richard's paper [34]. Our assumptions on the economy are quite general. In particular, preferences are non-ordered and we do not assume any monotonicity of preferences or free disposal in production. Under (most often) stronger assumptions on the characteristics of the economy, the equilibrium existence problem has been solved in two consecutive settings. In the first one, introduced by Mas-Colell [27], the commodity space is is a topological vector lattice. In the second one, introduced by Mas-Colell–Richard [29], the commodity space is an ordered locally convex topological vector space such that both commodity space and price space are lattice ordered. In these two settings, the equilibrium existence is obtained under properness assumptions on the characteristics of the economy. Since its first introduction to economics by Mas-Colell in [27] and [28], the formulation of properness has become more and more general [33, 23, 37, 38], but the lattice properties of the commodity–price duality are always used in a nontrivial way in the few papers [3, 39, 34, 38, 23] which address the equilibrium existence problem for production economies.

In this paper, we replace these lattice properties by a condition of compatibility between order and topology of the commodity space, stated in terms of properness of the Riesz–Kantorovich functional associated with a finite list of continuous linear functionals, which is obviously satisfied under these properties. The condition stated in this paper is stronger than the one used in our previous papers for the exchange case. Also, since the commodity space is not assumed to be a vector lattice, at least on the production side, properness has to be reformulated in a way which does not depend on lattice properties of the underlying space.

Our paper is organized as follows. In section 2, we recall the notion of the Riesz–Kantorovich functional associated with a finite list of (linear) functionals, a notion that we put in relation with the concept of sup-convolution, well-known inoptimization. In section 3, we define the model and posit assumptions as well on the commodity space as on the characteristics of the economy. The commodity space is assumed to be a locally convex topological vector space ordered by a closed and generating convex pointed cone such that the order intervals are topologically bounded. Mild, but classical, assumptions are made on the economy. Let us emphasize here that we do not assume any kind of monotonicity or transitivity on consumers' preferences or free disposal in production. Properness is defined as well for preferences as for production. Under a classical compactness assumption, Edgeworth equilibria exist. Section 4 is devoted to their decentralization with linear prices. For going from non-linear to linear decentralization, we introduce a condition of compatibility between order and topological structure of the commodity space stated in terms of uniform properness of the Riesz–Kantorovich functional. A by-product of the decentralization result is a quasiequilibium existence result. Existence of equilibrium is obtained under standard conditions of nontriviality and of irreducibility of the economy. In view of the generality of our assumptions, this result extends any previous equilibrium existence result obtained under properness in the Mas–Colell or in the Mas–Colell–Richard settings. Section 5 is devoted to examples which show that our result allows to obtain existence of equilibrium in settings of economic interest which are covered by no previous equilibrium existence result.

## 2. Mathematical preliminaries

For details regarding Riesz spaces that are not explained below we refer the reader to [6] and [7]. This paper will utilize the notion of the Riesz–Kantorovich formula that was introduced in [13] and used extensively in [14, 9, 10]. We shall briefly introduce this formula here and refer the reader to [12] for a complete discussion regarding the Riesz–Kantorovich formula.

We start with the following classical result from the theory of partially ordered vector spaces due to F. Riesz and L. V. Kantorovich.

**Theorem 2.1** (Riesz–Kantorovich)**.** *If $L$ is an ordered vector space with a generating cone and the Riesz Decomposition Property,*[1] *then the order dual $L^\sim$ is a Riesz space and for each $f, g \in L^\sim$ and $x \in L_+$ its lattice operations are given by:*

1. $[f \vee g](x) = \sup\big\{f(y) + g(z)\colon\ y, z \in L_+ \ \textit{ and } \ y + z = x\big\}$.
2. $[f \wedge g](x) = \inf\big\{f(y) + g(z)\colon\ y, z \in L_+ \ \textit{ and } \ y + z = x\big\}$ .

In particular, note that if $L$ has the Riesz Decomposition Property, then for any finite collection of linear functionals $f_1, f_2, \dots, f_m \in L^\sim$ their supremum in $L^\sim$ at each $x \in L_+$ is given by

$$\Big[\bigvee_{i=1}^m f_i\Big](x) = \sup\Big\{\sum_{i=1}^m f_i(x_i)\colon\ x_i \in L_+ \text{ for each } i \text{ and } \sum_{i=1}^m x_i = x\Big\}. \qquad (\star)$$

For any positive integer $m$ and $x \in L_+$ define

$$\mathcal{A}_x^m = \Big\{(x_1, \dots, x_m) \in L_+^m\colon\ \sum_{i=1}^m x_i = x\Big\}.$$

The formula $(\star)$ that gives the supremum of the order bounded linear functionals $f_1, \dots, f_m$ is called the Riesz–Kantorovich formula of these functionals. The useful observation here is that if each $f_i$ is an arbitrary function from $L_+$ to $(-\infty, \infty]$, then the right-hand side of $(\star)$ still defines an extended real number for each $x \in L_+$. That is, the formula appearing in $(\star)$ defines a function from $L_+$ to $(-\infty, \infty]$ called the **Riesz–Kantorovich functional** of the $m$-tuple of functions $f = (f_1, \dots, f_m)$ and denoted $\mathcal{R}_f$. In other words, $\mathcal{R}_f\colon L_+ \to (-\infty, \infty]$ is defined by

$$\mathcal{R}_f(x) := \sup\Big\{\sum_{i=1}^m f_i(x_i)\colon\ (x_1, \dots, x_m) \in \mathcal{A}_x^m\Big\}$$

for each $x \in L_+$.

If each $f_i$ is a function that carries order intervals to bounded from above subsets of $\mathbb{R}$, then the Riesz–Kantorovich functional is real-valued. Moreover, if each $f_i$ is super-additive and positively

[1] The Decomposition Property states that if $x$, $y_1$, $y_2 \in L_+$ satisfy $0 \le x \le y_1 + y_2$, then there exist elements $x_1$ and $x_2$ such that $0 \le x_1 \le y_1$, $0 \le x_2 \le y_2$ and $x = x_1 + x_2$. The order dual $L^\sim$ is the vector space consisting of all linear functionals on $L$ which map order intervals of $L$ to order bounded subsets of $\mathbb{R}$, ordered by the relation $f \ge g$ whenever $f(x) \ge g(x)$ for all $x \in L_+$.

homogeneous, then the Riesz–Kantorovich functional $\mathcal{R}_f$ is also super-additive and positively homogeneous; in particular, it is a concave function. Using as prices the Riesz–Kantorovich functional of a list of personalized prices, a new theory of value was presented in [14].

Let us now associate with each $f_i\colon L_+ \to (-\infty,\infty)$ the function $\widehat{f_i}\colon L \to [-\infty,\infty)$ defined by

$$\widehat{f_i}(x) := \begin{cases} f_i(x) & \text{if } x \in L_+ \\ -\infty & \text{otherwise} . \end{cases}$$

If each $f_i$ is finite-valued, it is easy to recognize in $\mathcal{R}_f(x)$ for $x \in L_+$ the value at $x$ of the **sup-convolution**[2] of functions $\widehat{f_i}$ defined by

$$\big[\nabla_{i=1}^m \widehat{f_i}\big](x) := \sup\Big\{ \sum_{i=1}^m \widehat{f_i}(x_i)\colon \ \sum_{i=1}^m x_i = x \Big\}.$$

**Definition 2.2.** *We will say that $\mathcal{R}_f$ is **exact** at $x$ with respect to a vector $(x_1,\dots,x_m) \in L_+^m$ satisfying $x = \sum_{i=1}^m x_i$, if the sup-convolution is exact at $x$ with respect to $(x_1,\dots,x_m) \in L_+^m$. That is, if*

$$\mathcal{R}_f(x) = \big[\nabla_{i=1}^m \widehat{f_i}\big](x) = \sum_{i=1}^m \widehat{f_i}(x_i) = \sum_{i=1}^m f_i(x_i) .$$

Let $\langle X, X'\rangle$ be a dual system and let $f \in \overline{\mathbb{R}}^X$. Recall that a vector $y' \in X'$ is called a **supergradient** of $f$ at $x$ if $f(x)$ is finite and $f(y) - f(x) \le \langle y - x, y'\rangle$ for all $y \in X$. The (possibly empty) set of all supergradients of $f$ at $x$ is called the **superdifferential** at $x$ of the function $f$ and denoted $\partial f(x)$.

The following result (see for example [26, Proposition 6.6.4]) will be used in our work.

**Theorem 2.3** (Moreau)**.** *Assume that $\langle X, X'\rangle$ is an arbitrary dual system. For each $i = 1,\dots,m$ let $g_i\colon X \to [-\infty,\infty]$ be a non identically equal to $-\infty$ function. If the sup-convolution $\nabla_{i=1}^m g_i$ is exact at $x$ with respect to some $(x_1,\dots,x_m) \in X^m$ that satisfies $x = \sum_{i=1}^m x_i$, then*

$$\partial\big[\nabla_{i=1}^m g_i\big](x) = \bigcap_{i=1}^m \partial g_i(x_i).$$

## 3. The economic model

In what follows, if $M$ is an ordered linear space, then for the sake of notational convenience, $M^+$ or $M_+$ will denote the positive cone of $M$.

The **commodity space** of our model is an ordered linear vector space $L$ equipped with a Hausdorff locally convex topology $\tau$ such that:

**A1:** *The positive cone $L_+$ of $L$ is generating (i.e., $L = L_+ - L_+$) and $\tau$-closed.*
**A2:** *The order intervals of $L$ are $\tau$-bounded.*

[2] Let $f \in \overline{\mathbb{R}}^L$ and $g \in \overline{\mathbb{R}}^L$ be extended real-valued functions. Using the convention $+\infty + (-\infty) = -\infty + (+\infty) = -\infty$, the formula $[f\nabla g](x) = \sup\{f(y) + g(z)\colon \ y,z \in L \ \text{ and } \ y + z = x\}$ defines an extended real-valued function $f\nabla g$ called the *sup-convolution* of $f$ and $g$. The expression $f\nabla g$ is also called by Rockafellar and Wets [35] the *hypo-addition* of functions $f$ and $g$, because if hypo $f$ denotes the hypograph of $f$, one has $\operatorname{hypo}(f\nabla g) = \operatorname{hypo} f + \operatorname{hypo} g$, as long as the supremum defining $[f\nabla g](x)$ is attained when finite.

The *topological dual* of $(L, \tau)$ (i.e., the vector space of all $\tau$-continuous linear functionals on $L$) will be denoted $L'$. The *algebraic dual* of $L$ (i.e., the vector space of all linear functionals on $L$) is denoted $L^*$. The *order dual* of $L$ (i.e., the vector space of all order-bounded linear functionals on $L$) is denoted $L^\sim$. Since every order interval of $L$ is $\tau$-bounded, it follows that $L' \subseteq L^\sim \subseteq L^*$.

Assumptions **A1** and **A2** are satisfied by the following ones used by Mas-Colell–Richard [29] and many others:

**B1:** *$L$ is a vector lattice whose positive cone $L_+$ is $\tau$-closed; and*
**B2:** *$L'$ is a vector sublattice of $L^\sim$.*

Both sets of assumptions are implied by Mas-Colell's assumption in [27] and [28] that $L$ is a topological vector lattice. To see this, note that if $L$ is a topological vector lattice, then **B1** simply follows from the continuity of lattice operations, and **B2** follows from the Nakano–Roberts theorem (see [7, Theorem 2.22]). On the other hand, **B2** implies that the cone $L'_+$ is generating, which is equivalent the fact that the cone $L_+$ is a normal cone for the weak topology $\sigma(L, L')$, and the latter implies that the order intervals are $\sigma(L, L')$-bounded, thus $\tau$-bounded (see [32, Chapter 2, Corollary 1.23 and Proposition 1.4, and 2.1 for the definition of normal cones]). The assumption that $L$ is a vector lattice and **B1** obviously imply **A1**.

On $L$ as commodity space, let us consider a private ownership production economy

$$\mathcal{E} = \Big( \big(X_i, P_i, \omega_i\big)_{i \in I}, (Y_j)_{j \in J}, (\theta_{ij})_{i \in I, j \in J} \Big)$$

where $I = \{1, \dots, m\}$ is a finite set of $m\,(\geq 2)$ consumers, $J = \{1, \dots, n\}$ is a finite set of $n\,(\geq 1)$ producers. Each consumer $i$ is characterized by a non-empty **consumption set** $X_i \subseteq L$, an **initial endowment** $\omega_i \in X_i$ and an irreflexive **preference correspondence** $P_i \colon X_i \twoheadrightarrow X_i$, i.e., $x_i \notin P_i(x_i)$ for each $x_i \in X_i$. Each producer $j$ is characterized by a non-empty **production set** $Y_j \subseteq L$. For every producer and each consumer, the firm shares $0 \leq \theta_{ij} \leq 1$ classically represent a contractual claim of consumer $i$ to the profit of producer $j$ and $\sum_{i \in I} \theta_{ij} = 1$ for each $j \in J$. In a core and Edgeworth equilibrium approach, the relative shares $\theta_{ij}$ reflect consumers' stockholdings that represent proprietorships of production possibilities and $\theta_{ij} Y_j$ is the portion of the $j$ producer's technology set at $i$'s disposal.

Let $\omega = \sum_{i \in I} \omega_i$ be the **total endowment**, and let $\mathcal{A}_\omega$ be the set of all **feasible (or attainable) allocations** of $\mathcal{E}$, that is,

$$\mathcal{A}_\omega = \Big\{ (x, y) = \big((x_i)_{i \in I}, (y_j)_{j \in J}\big) \in \prod_{i \in I} X_i \times \prod_{j \in J} Y_j \colon \sum_{i \in I} x_i = \omega + \sum_{j \in J} y_j \Big\}.$$

The set $X_\omega$, the projection of $\mathcal{A}_\omega$ on $\prod_{i \in I} X_i$,[3] is the set of all **feasible consumption allocations**.

We recall the following standard notions of equilibria for an economy $\mathcal{E}$.

**Definition 3.1.** *A 3-tuple $(\overline{x}, \overline{y}, \overline{p})$ consisting of a feasible allocation $(\overline{x}, \overline{y})$ and a non-zero linear functional $\overline{p}$ is said to be:*

1. *a **quasi-equilibrium**, if*
   (a) *for every $i \in I$ we have $\overline{p}(\overline{x}_i) = \overline{p}(\omega_i) + \sum_{j \in J} \theta_{ij} \overline{p}(\overline{y}_j)$ and $x_i \in P_i(\overline{x}_i)$ implies $\overline{p}(x_i) \geq \overline{p}(\overline{x}_i)$, and*
   (b) *for every $j \in J$ and every $y_j \in Y_j$ we have $\overline{p}(y_j) \leq \overline{p}(\overline{y}_j)$;*
2. *an **equilibrium**, if it is a quasi-equilibrium and if $x_i \in P_i(\overline{x}_i)$ implies $\overline{p}(x_i) > p(\overline{x}_i)$.*

[3] That is, $X_\omega = \{x = (x_i)_{i \in I} \in \prod_{i \in I} X_i \colon \ \sum_{i \in I} x_i = \omega + \sum_{j \in J} y_j$ for some $(y_j)_{j \in J} \in \prod_{j \in J} Y_j\}$.

**Definition 3.2.** *A quasi-equilibrium* $(\overline{x}, \overline{y}, \overline{p})$ *is said to be* ***nontrivial****, if for some* $i \in I$ *we have*

$$\inf\{\overline{p}(z_i)\colon\ z_i \in X_i\} < \overline{p}(\overline{x}_i).$$

In this paper, we will be interested only in nontrivial quasi-equilibria. If $(\overline{x}, \overline{y}, \overline{p})$ is some trivial quasi-equilibrium, then for every feasible allocation $(x, y)$, the pair $\big((x, y), \overline{p}\big)$ is also a quasi-equilibrium. On the other hand, if the quasi-equilibrium $(\overline{x}, \overline{y}, \overline{p})$ is nontrivial, then it is well-known that, under some continuity condition on preferences or concavity for utility functions, and some irreducibility assumption on the economy, the quasi-equilibrium $(\overline{x}, \overline{y}, \overline{p})$ is actually an equilibrium.

The following notions of optimality are also standard.

**Definition 3.3.** *A (feasible) consumption allocation* $\overline{x} \in X_\omega$ *is said to be:*

1. ***weakly Pareto optimal****, if there is no feasible consumption allocation* $x \in X_\omega$ *satisfying* $x_i \in P_i(\overline{x}_i)$ *for each* $i \in I$*,*
2. *a* ***core allocation****, if it cannot be blocked by any coalition in the sense that there is no coalition* $S \subseteq I$ *and some* $x \in \prod_{i\in S} X_i$ *such that:*
   (a) $\sum_{i\in S} x_i \in \sum_{i\in S} \omega_i + \sum_{i\in S} \sum_{j\in J} \theta_{ij} Y_j$*, and*
   (b) $x_i \in P_i(\overline{x}_i)$ *for all* $i \in S$*,*
3. *an* ***Edgeworth equilibrium****, if it belongs to the core of every* $r$*-fold replica of* $\mathcal{E}$*,*[4]
4. *a* ***fuzzy core allocation****, if there exist no* $\tau = (\tau_i)_{i\in I} \in [0, 1]^I \setminus \{0\}$ *and* $x \in \prod_{i\in I} X_i$ *such that:*
   (a) $\sum_{i\in I} \tau_i x_i \in \sum_{i\in I} \tau_i \omega_i + \sum_{i\in I} \tau_i \sum_{j\in J} \theta_{ij} Y_j$*, and*
   (b) $x_i \in P_i(\overline{x}_i)$ *for all* $i \in I$ *with* $\tau_i > 0$*.*

From now on, we impose on $\mathcal{E}$ the following conditions.

$\omega$**-properness:** That is,

1. For each consumer $i$, the consumption set $X_i$ is convex, and $\omega = \sum_{i\in I} \omega_i > 0$.
2. For each $i$ and every weakly Pareto optimal consumption allocation $x = (x_i)_{i\in I}$, $x_i \in \operatorname{cl} P_i(x_i)$, $P_i(x_i)$ is open in $X_i$ for some linear topology on $L$ (or is induced by a concave utility function) and is $\omega$**-proper** at $x_i$ in the following sense adapted from Tourky [37]:
   there exist in $L$ a convex set $\widehat{P}_i(x_i)$ and a convex set $Z_i(x_i)$ such that
   (a) the vector $x_i + \omega$ is a $\tau$-interior point of $\widehat{P}_i(x_i)$,
   (b) $\widehat{P}_i(x_i) \cap Z_i(x_i) = P_i(x_i)$, and
   (c) $x_i$, $0$, $\omega_i \in Z_i(x_i)$, $Z_i(x_i) + L_+ \subseteq Z_i(x_i)$,
   (d) for every $u > 0$ and every couple $\{z_i, z_i'\}$ of elements of $Z_i(x_i)$, if $-u \leq z_i$, $-u \leq z_i'$ then there exists $z \in Z_i(x_i)$ such that $-u \leq z \leq z_i$ and $-u \leq z \leq z_i'$.
3. For each producer $j$, the production set $Y_j$ is convex and $0 \in Y_j$ .
4. For each $j$ and every $y_j \in Y_j$ associated with a weakly Pareto optimal consumption allocation, $Y_j$ is $\omega$**-proper at** $y_j$ in the following sense adapted from Tourky [38]:
   there exist in $L$ a convex set $\widehat{Y}_j(y_j)$ and a convex set $Z_j(y_j)$ such that
   (a) the vector $y_j - \omega$ is a $\tau$-interior point of $\widehat{Y}_j(y_j)$,
   (b) $\widehat{Y}_j(y_j) \cap Z_j(y_j) = Y_j$, and
   (c) $0 \in Z_j(y_j)$ and $Z_j(y_j) - L_+ \subseteq Z_j(y_j)$,

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[4] The ideas in this definition go back to Debreu–Scarf [19]. An important reference is also [17]. Edgeworth equilibria were first introduced and studied in [2].

(d) for every $u > 0$ and every couple $\{z_j, z'_j\}$ of elements of $Z_j(y_j)$, if $z_j \leq u$, $z'_j \leq u$ then there exists $z \in Z_j(y_j)$ such that $z_j \leq z \leq u$ and $z'_j \leq z \leq u$.

**Compactness:** For some Hausdorff linear topology $\sigma$ on $L$, the set $X_\omega$ of all feasible consumption allocations is $\sigma^m$-compact and preferences have $\sigma$-open (in $X_i$) lower sections $P_i^{-1}(x_i) = \{x'_i \in X_i\colon\ x_i \in P_i(x'_i)\}$.

At this stage, some comments on our properness assumptions are in order.

In our hypothesis, $\omega$-properness of preferences at a component of a (feasible) weakly Pareto optimal consumption allocation is stated in a general form for the sake of generality and symmetry with our definition of $\omega$-properness of production sets. In the classical case, the consumption sets $X_i$ and the sets $Z_i(x_i)$ all coincide with the positive cone $L_+$ of the commodity space $L$ which obviously satisfies the assumptions required for the sets $Z_i(x_i)$. The interest of our definition is to allow for more general consumption sets. It is worth noticing that if the commodity space $L$ is a Riesz space, then our assumptions on the sets $Z_i(x_i)$ are equivalent to Tourky's assumption in [37] that each $Z_i(x_i)$ is a convex lattice containing 0, $\omega_i$, $x_i$ and satisfying $Z_i(x_i) + L_+ \subseteq Z_i(x_i)$. Let us stress that our assumptions do not imply that $L_+$ is a lattice cone.

The sets $Z_j(y_j)$ which are involved in the definition of $\omega$-properness of production sets at a production component of a weakly Pareto optimal allocation correspond to the pretechnology sets defined by Mas-Colell [28] and used by Richard [34]. In Mas-Colell [28] and Richard [34], the pretechnology sets depend on $j$ but not on a particular $y_j$. As above, if the commodity space $L$ is a Riesz space, then our assumptions on the sets $Z_j(y_j)$ are equivalent to Tourky's assumption in [38] that each $Z_j(y_j)$ is a convex lattice containing 0, $y_j$ and satisfying $Z_j(y_j) - L_+ \subseteq Z_j(y_j)$. When the commodity space is an ordered vector space but not a Riesz space, then pretechnology sets equal to the negative cone $-L_+$ satisfy obviously our assumptions but are not the only one such examples.

To conclude this section, let us remark that it easily follows from the previous definitions that every equilibrium consumption allocation is an element of the fuzzy core and, consequently, an Edgeworth equilibrium, a core allocation and a weakly Pareto optimal allocation. The following lemma follows from the previous assumptions and Florenzano [21].

**Lemma 3.4.** *In our economy, Edgeworth equilibria exist and belong to the fuzzy core.*

Exactly as in [15], one can prove that if preference correspondences are derived from quasiconcave utility functions defined on general consumption sets, our compactness condition can be replaced by a weaker compactness assumption made in the utility space on the "utility set".[5]

The next section is devoted to look at sufficient conditions on the order structure of the commodity space that will guarantee for our economy the validity of the core equivalence theorem (first stated by Debreu–Scarf [19] for a finite dimensional economy).

## 4. Decentralizing Edgeworth equilibria

For the discussion of this section, we fix an Edgeworth equilibrium consumption allocation $(\overline{x}_i)_{i\in I}$, thus a fuzzy core consumption allocation of $\mathcal{E}$, and $(\overline{y}_j)_{j\in J}$, the associated production

[5] That is, the set of feasible and individually rational utility vectors.

allocation. Such an allocation $\big((\overline{x}_i)_{i\in I}, (\overline{y}_j)_{j\in J}\big)$ exists by Lemma 3.4. The proof of its possible decentralization by prices in $L'$ will adapt ideas of [22, Chapter 5.3.4] where is proved the decentralization of Edgeworth equilibria of an $\omega$-proper production economy defined on a vector lattice commodity space satisfying **B1** and **B2**. For technical details not covered in this paper, we refer to this chapter.

Since $L_+$ is by **A1** generating, we can choose $u > 0$ in $L$ such that the order interval $[-u, u]$ contains $\overline{x}_i, \omega_i, \overline{y}_j$ for all $i \in I$ and $j \in J$. Now consider the ordered vector subspace $L_u = \bigcup_{\lambda>0} \lambda[-u, u]$ equipped with the order topology. This space is Archimedean (a property inherited from $L$ of which the positive cone is $\tau$-closed) and has $u$ as an order unit.[6] Its order topology (i.e., the finest locally convex topology on $L_u$ for which every order interval is bounded) is normable (see Schaefer [36, Chapter V 6.2]). More precisely, the gauge $\|\cdot\|_u$ of $[-u, u]$, defined for each $z \in L_u$ by

$$\|z\|_u = \inf\big\{\lambda > 0\colon\ -\lambda u \le z \le \lambda u\big\}\,,$$

is a norm on $L_u$ that generates the order topology whose closed unit ball is precisely the order interval $[-u, u]$ and $u$ is an interior point of $L_u^+ = L_u \cap L_+$. Moreover, it follows from **A2** (namely from the fact that $[-u, u]$ is $\tau$-bounded) that on $L_u$, the order topology is finer than the topology induced by $\tau$. In addition, it is not difficult to see that the cone $L_u^+$ is $\|\cdot\|_u$-closed in $L_u$.[7]

Let $\mathcal{E}_u$ be the economy $\mathcal{E}$ restricted to $L_u$ in an obvious way. For each consumer $i$, $X_i^u = X_i \cap L_u$, and for each $x_i \in X_i^u$, $P_i^u(x_i) = P_i(x_i) \cap L_u$; for each producer $j$, $Y_j^u = Y_j \cap L_u$. It is easily seen that the consumption allocation $(\overline{x}_i)_{i\in I}$ is an Edgeworth equilibrium and belongs to the fuzzy core of $\mathcal{E}_u$. It follows from the $\omega$-properness of $\mathcal{E}$ that for each $i$ and any $0 < \alpha \le 1$, the vector $\overline{x}_i + \alpha\omega$ belongs to the norm-interior of $P_i^u(\overline{x}_i)$. It thus follows from the standard decentralization result in presence of an interiority property that there exists a nonzero $\overline{p}_u \in (L_u, \|\cdot\|_u)'$ such that $(\overline{x}, \overline{y}, \overline{p}_u)$ is a quasi-equilibrium of the restricted economy $\mathcal{E}_u$ and $\overline{p}_u(\omega) > 0$.

Before going further, let us recall an extension lemma due to Podczeck [33] whose a proof can be found in [22, Lemma 5.3.1, p. 134]. In [33], this lemma is used for deducing the existence of equilibria for a proper exchange economy defined on a vector lattice commodity space satisfying **B1** and **B2** from the equilibrium existence in the economy restricted to the order ideal generated by the total endowment.

**Lemma 4.1** (Podczeck)**.** *Let $(L, \tau)$ be an ordered topological vector space, let $K$ be a vector subspace of $L$ (endowed with the induced order), let $A$ be a convex subset of $K$ such that $A + K_+ \subseteq A$, let $V$ be a convex $\tau$-open subset of $L$ such that $V \cap A \neq \emptyset$, and let $a \in A \cap \operatorname{cl} V$. If $p$ is a linear functional on $K$ satisfying*

$$p \cdot a \le p \cdot x, \text{ for all } x \in V \cap A,$$

*then there exists some $\pi \in L'$ such that $\pi_{|K} \le p$, and*

$$p \cdot (a - x) = \pi \cdot (a - x) \text{ for each } x \in A \text{ with } x \le a\,.$$

The following obvious consequence of the preceding lemma was stated in [23].

[6] That is, the order interval $[-u, u]$ is radial at the origin.

[7] Indeed, if $\{x_n\} \subseteq L_+$ satisfies $-\frac{1}{n}u \le x - x_n \le \frac{1}{n}u$ for all $n$, then from $x = (x - x_n) + x_n \ge x - x_n \ge -\frac{1}{n}u$ and the Archimedean property it follows that $x \ge 0$.

**Corollary 4.2.** *Let $(L,\tau)$ be an ordered topological vector space , let $K$ be a vector subspace of $L$ (endowed with the induced order), let $A$ be a convex subset of $K$ such that $A - K_+ \subseteq A$, let $V$ be a convex $\tau$-open subset of $L$ such that $V \cap A \neq \emptyset$, and let $a \in A \cap \operatorname{cl} V$. If $p$ is a linear functional on $K$ satisfying*

$$p \cdot a \geq p \cdot x, \text{ for all } y \in V \cap A\,,$$

*then there exists some $\pi \in L'$ such that $\pi_{|K} \leq p$, and*

$$p \cdot (a - y) = \pi \cdot (a - y) \text{ for each } y \geq a,\ y \in A\,.$$

Our next step associates with the quasi-equilibrium price $\overline{p}_u$ a finite list of $\tau$-continuous linear functionals $\pi^u = \big((\pi^u_i)_{i\in I}, (\pi^u_j)_{j\in J}\big)$ defined on the whole commodity space. These continuous linear functionals can be thought of as personalized supporting prices for each consumer and each producer at the corresponding component of the allocation $(\overline{x}, \overline{y})$.

**Proposition 4.1.** *In our economy, there exist $(\pi^u_i)_{i\in I}$, $(\pi^u_j)_{j\in J}$ in $L'$ such that:*

1. *For each consumer $i$ we have*
   (a) *$\pi^u_i \leq \overline{p}_u$ on $L_u$,*
   (b) *$\pi^u_i \cdot \widehat{P}_i(\overline{x}_i) \geq \pi^u_i \cdot \overline{x}_i$ , and*
   (c) *if $z_i \in Z_i(\overline{x}_i) \cap L_u$ satisfies $z_i \leq \overline{x}_i$, then*

$$\overline{p}_u \cdot (\overline{x}_i - z_i) = \pi^u_i \cdot (\overline{x}_i - z_i) = \mathcal{R}_{\pi^u}(\overline{x}_i - z_i)\,.$$

2. *For each producer $j$ we have*
   (a) *$\pi^u_j \leq \overline{p}_u$ on $L_u$,*
   (b) *$\pi^u_j \cdot \widehat{Y}_j(\overline{y}_j) \leq \pi^u_j \cdot \overline{y}_j$ , and*
   (c) *if $z_j \in Z_j(\overline{y}_j) \cap L_u$ satisfies $z_j \geq \overline{y}_j$ , then*

$$\overline{p}_u \cdot (z_j - \overline{y}_j) = \pi^u_j \cdot (z_j - \overline{y}_j) = \mathcal{R}_{\pi^u}(z_j - \overline{y}_j)\,.$$

3. *Moreover,*
   (a) *$\mathcal{R}_{\pi^u} \leq \overline{p}_u$ on $L_u$, and*
   (b) *if $v \leq \omega$ satisfies $v = \sum_{i\in I} v_i - \sum_{j\in J} w_j \in \Big[\sum_{i\in I}\big(Z_i(\overline{x}_i) \cap L_u\big) - \sum_{j\in J}\big(Z_j(\overline{y}_j) \cap L_u\big)\Big]$, then*

$$\overline{p}_u(\omega - v) = \mathcal{R}_{\pi^u}(\omega - v)\,.$$

*Proof.* Using Lemma 4.1 and its corollary with $K = L_u$ and for each $i$, $a = \overline{x}_i$, $A = Z_i(\overline{x}_i) \cap L_u$, $V = \operatorname{int} \widehat{P}_i(\overline{x}_i)$, for each $j$, $a = \overline{y}_j$, $A = Z_j(\overline{y}_j) \cap L_u$, $V = \operatorname{int} \widehat{Y}_j(\overline{y}_j)$, the existence of $(\pi^u_i)_{i\in I}$, $(\pi^u_j)_{j\in J}$ in $L'$ satisfying (1), (2) and (3)(a) is a straightforward consequence of the fact that $(\overline{x}, \overline{y}, \overline{p}_u)$ is a quasi-equilibrium of the restricted economy $\mathcal{E}_u$ and of the $\omega$-properness assumptions at each component of the allocation $(\overline{x}, \overline{y})$.

To prove (3)(b), assume that $v \leq \omega$, $v = \sum_{i\in I} v_i - \sum_{j\in J} w_j$ with each $v_i \in Z_i(\overline{x}_i) \cap L_u$ and each $w_j \in Z_j(\overline{y}_j) \cap L_u$. Let $z_i \in Z_i(\overline{x}_i) \cap L_u$ be such that $z_i \leq \{v_i, \overline{x}_i\}$ and $z_j \in Z_j(y_j) \cap L_u$ be such that $z_j \geq \{w_j, \overline{y}_j\}$. Such $z_i$ and $z_j$ exist in view of the definition of $L_u$ and our assumptions on sets $Z_i(\overline{x}_i)$ and $Z_j(\overline{y}_j)$. On one hand, using (1)(c), (2)(c) and the superadditivity of $\mathcal{R}_{\pi^u}$, it follows from (1)(a) and (2)(a) that

$$\begin{aligned}(\overline{p}_u - \mathcal{R}_{\pi^u})(\omega - v) &= (\overline{p}_u - \mathcal{R}_{\pi^u})\Big(\sum_{i\in I}\overline{x}_i - \sum_{j\in J}\overline{y}_j - \sum_{i\in I} v_i + \sum_{j\in J} w_j\Big)\\ &\leq (\overline{p}_u - \mathcal{R}_{\pi^u})\Big(\sum_{i\in I}(\overline{x}_i - z_i) + \sum_{j\in J}(z_j - \overline{y}_j)\Big) = 0\,.\end{aligned}$$

On the other hand, it follows also from (1)(a) and (2)(a) that $(\overline{p}_u - \mathcal{R}_{\pi^u})(\omega - v) \geq 0$, which implies $(\overline{p}_u - \mathcal{R}_{\pi^u})(\omega - v) = 0$ ■

The following corollary can be seen as an analogue for the production economy $\mathcal{E}_u$ of Proposition 5.1 in [9] which extends at several instances statement (1) of Theorem 7.5 in [14] .

**Corollary 4.3.** *We have in addition:*

1. $x_i \in P_i^u(\overline{x}_i) \Longrightarrow \mathcal{R}_{\pi_u}(x_i - z_i) \geq \mathcal{R}_{\pi_u}(\overline{x}_i - z_i)$ *for every* $z_i \in Z_i(\overline{x}_i) \cap L_u$, $z_i \leq \{x_i\,, \overline{x}_i\,\}$.
2. $y_j \in Y_j^u \Longrightarrow \mathcal{R}_{\pi_u}(z_j - y_j) \geq \mathcal{R}_{\pi_u}(z_j - \overline{y}_j)$ *for every* $z_j \in Z_j(\overline{y}_j) \cap L_u$, $z_j \geq \{y_j\,, \overline{y}_j\,\}$.
3. *If for each* $i$, $z_i' \in Z_i(\overline{x}_i) \cap L_u$, $z_i' \leq \{\omega_i\,, \overline{x}_i\}$, *if for each* $j$, $z_j' \in Z_j(\overline{y}_j) \cap L_u$, $z_j' \geq \{0, \overline{y}_j\}$, *then*
   (a) $\mathcal{R}_{\pi_u}(\omega - \sum_{i\in I} z_i' + \sum_{j\in J} z_j') = \sum_{i\in I}\pi_i^u(\overline{x}_i - z_i') + \sum_{j\in J}\pi_j^u(z_j' - \overline{y}_j)$, *and*
   (b) *for each* $i$, $\pi_i^u \cdot (\overline{x}_i - z_i') + \sum_{j\in J}\theta_{ij}\pi_j^u \cdot (z_j' - \overline{y}_j) \geq \mathcal{R}_{\pi_u}(\omega_i - z_i' + \sum_{j\in J}\theta_{ij} z_j')$.

*Proof.* To prove (1), let us assume $x_i \in P_i^u(\overline{x}_i)$ and $z_i \in Z_i(\overline{x}_i) \cap L_u, z_i \leq \{x_i\,, \overline{x}_i\,\}$. Recall that $P_i(\overline{x}_i) = \widehat{P}_i(\overline{x}_i) \cap Z_i(\overline{x}_i)$, thus that $P_i^u(\overline{x}_i) = \widehat{P}_i(\overline{x}_i) \cap Z_i(\overline{x}_i) \cap L_u$. We then easily deduce from Proposition 4.1 that:

$$\mathcal{R}_{\pi_u}(x_i - z_i) \geq \pi_i^u \cdot (x_i - z_i) \geq \pi_i^u \cdot (\overline{x}_i - z_i) = \mathcal{R}_{\pi_u}(\overline{x}_i - z_i).$$

The proof of (2) is done symmetrically. The proof of (3) goes as follows.

Using the superadditivity of $\mathcal{R}_{\pi_u}$, the last assertion of Proposition 4.1, and assuming that for each $i$, $z_i' \in Z_i(\overline{x}_i) \cap L_u$, $z_i' \leq \{\omega_i\,, \overline{x}_i\}$, and for each $j$, $z_j' \in Z_j(\overline{y}_j) \cap L_u$, $z_j' \geq \{0, \overline{y}_j\}$,

we get:

$$\overline{p}_u(\omega - \sum_{i\in I} z_i' + \sum_{j\in J} z_j') = \mathcal{R}_{\pi_u}(\omega - \sum_{i\in I} z_i' + \sum_{j\in J} z_j') \geq \sum_{i\in I}\mathcal{R}_{\pi^u}(\overline{x}_i - z_i') + \sum_{j\in J}\mathcal{R}_{\pi^u}(z_j' - \overline{y}_j)$$

$$= \sum_{i\in I}\pi_i^u(\overline{x}_i - z_i') + \sum_{j\in J}\pi_j^u(z_j' - \overline{y}_j) = \overline{p}_u(\omega - \sum_{i\in I} z_i' + \sum_{j\in J} z_j'),$$

which proves the first assertion of (3).

Finally, recall that $(\overline{x}, \overline{y}, \overline{p}_u)$ is a quasi-equilibrium of $\mathcal{E}_u$. We thus have for every $i$,

$$\overline{p}_u \cdot \overline{x}_i = \overline{p}_u \cdot \omega_i + \sum_{j\in J}\theta_{ij}\overline{p}_u \cdot \overline{y}_j. \tag{4.1}$$

From (4.1), using (1)(c) and (2)(c) of Proposition 4.1, we deduce easily from that for each $i$,

$$\mathcal{R}_{\pi_u}(\overline{x}_i - z_i') + \sum_{j\in J}\theta_{ij}\mathcal{R}_{\pi_u}(z_j' - \overline{y}_j) = \overline{p}_u \cdot (\overline{x}_i - z_i') + \sum_{j\in J}\theta_{ij}\overline{p}_u \cdot (z_j' - \overline{y}_j)$$

$$= \overline{p}_u \cdot (\omega_i - z_i') + \sum_{j\in J}\theta_{ij}\overline{p}_u \cdot z_j' \geq \mathcal{R}_{\pi_u}\big(\omega_i - z_i' + \sum_{j\in J}\theta_{ij} z_j'\big),$$

which completes the proof. ■

Let us now recall some notions of properness for functions on $L_+$.

**Definition 4.4.** *Let $v \in L_+$ be such that $v > 0$. We say that a function $f\colon L_+ \to \mathbb{R}$ is:*

1. *$v$-**proper** at some $x \in L_+$, if there exists a convex set $F$ such that:*
   (a) *$x + v$ is an interior point of $F$, and*
   (b) *$F \cap L_+ = \{y \in L_+ \colon\ f(y) > f(x)\}$.*
2. *$v$-**pointwise proper** at $x \in L_+$, if there exists an open pointed convex cone $\Gamma_x$ such that:*
   (a) *$-v \in \Gamma_x$, and*
   (b) *$(x - \Gamma_x) \cap \big\{y \in L_+ \colon\ f(y) > f(x)\big\} = \emptyset$.*

Considering the set $\{y \in L_+ \colon\ f(y) > f(x)\}$ as a preferred set for a preference correspondence defined on $L_+$ by the utility function $f$, the reader recognizes in these definitions the usual notions of $v$-pointwise properness as defined by Mas-Colell [27] and of $v$- properness as defined by Tourky [37]. Mas-Colell [27] defines uniform properness on a subset $X$ of $L_+$ as properness at every $x \in X$ with a properness vector and a properness cone independent of $x$.[8] The definition of uniform $v$-properness given here is quite similar.

**Definition 4.5.** *The function $f\colon L_+ \to \mathbb{R}$ is **uniformly** $v$-**proper** on $X \subset L_+$ if for every $x \in X$ there exist a convex set $F_x$ and a $\tau$-neighborhood $V$ of $0$ (independent of $x$) such that*

a. *$x + v + V \subseteq F_x$, and*
b. *$F_x \cap L_+ = \{y \in L_+ \colon\ f(y) > f(x)\}$.*

We now introduce the following additional assumption of compatibility between the order structure and the topology $\tau$ of the commodity space of our economy.

**A3:** *For any finite list $f$ of continuous linear functionals $(f_k)_{k=1}^{K}$ such that $f_k(\omega) > 0$ for each $k$, the Riesz–Kantorovich functional $\mathcal{R}_f$ is uniformly $\omega$-proper at any point $\omega' \geq \omega$, provided that it is exact at this point.*

**Remark 4.6.** This hypothesis is automatically satisfied under the assumptions **B1** and **B2**, and a fortiori if the commodity space is a locally solid Riesz space. Indeed, it follows from Theorem 2.1 and Assumptions **B1**, **B2** that the Riesz–Kantorovich functionals are linear and $\tau$-continuous. Assume now that $f$ is a finite list of continuous linear functionals $(f_k)_{k=1}^{K}$ such that $f_k(\omega) > 0$ for each $k$ (which implies $\mathcal{R}_f(\omega) > 0$). For each $\omega' \in L_+$, define $F_{\omega'} = \{y \in L \colon\ g(y) > g(\omega')\}$, where $g$ is the continuous linear functional which coincides with $\mathcal{R}_f$ on $L_+$. If $V$ is a $\tau$-neighborhood of $0$ such that $g(\omega + V) > 0$, then $g(\omega' + \omega + V) > g(\omega')$, thus $\omega' + \omega + V \subseteq F_{\omega'}$, while $F_{\omega'} \cap L_+ = \{y \in L \colon\ \mathcal{R}_f(y) > \mathcal{R}_f(\omega')\}$.

**Proposition 4.2.** *Under **A3**, there exists a price system $\overline{\pi}_u \in L'$ such that $\overline{\pi}_u{\cdot}\omega > 0$ and $(\overline{x}, \overline{y}, \overline{\pi}_u)$ is a quasi-equilibrium of $\mathcal{E}_u$.*

*Proof.* Let us consider the set $\Omega$ of all $\omega' = \omega - \sum_{i\in I} z_i' + \sum_{j\in J} z_j'$, where for each $i$, $z_i' \in Z_i(\overline{x}_i) \cap L_u$, $z_i' \leq \{0, \omega_i\,, \overline{x}_i\}$, and for each $j$, $z_j' \in Z_j(\overline{y}_j) \cap L_u$, $z_j' \geq \{0, \overline{y}_j\}$. This set is nonempty in view of the definition of $L_u$ and our assumptions on $Z_i(\overline{x}_i)$ and $Z_j(\overline{y}_j)$. It is directed by the relation

$$\omega'' \geq \omega' \ \text{ if and only if } \ \text{for each } \ i \ \text{ and } \ j,\ z_i'' \leq z_i' \ \text{ and } z_j'' \geq z_j'.$$

[8] If $v$ is the properness vector, recall that Mas-Colell's uniform properness on $L_+$ implies $v$-properness (see Tourky [37]).

Moreover, if $\omega' \in \Omega$ then $z'_i \leq 0$ for each $i$ and $z'_j \geq 0$ for each $j$ imply $\omega' \geq \omega > 0$. From (3) in Corollary 4.3, it follows that the Riesz–Kantorovich functional $\mathcal{R}_{\pi_u}$ is exact at each $\omega' \in \Omega$. From Proposition 4.1, it is easily deduced that for each $i$, either $\pi_i^u = 0$ or $\pi_i^u \cdot \omega > 0$, that for each $j$, either $\pi_j^u = 0$ or $\pi_j^u \cdot \omega > 0$, that $\mathcal{R}_{\pi^u}(\omega) = \overline{p}_u(\omega) > 0$, thus that at least one of the $\pi_i^u$, $\pi_j^u$ is nonzero. Set $P(\omega') = \{z \in L_+ \colon\ \mathcal{R}_{\pi_u}(z) > \mathcal{R}_{\pi_u}(\omega')\}$. Applying Assumption **A3**, there exist a 0-neighborhood $V$ and for each $\omega' \in \Omega$, a convex set $\widehat{P}(\omega')$ such that $\omega' + \omega + V \subseteq \widehat{P}(\omega')$ and $P(\omega') = L_+ \cap \widehat{P}(\omega')$.

Moreover,

$$\mathcal{R}_{\pi_u}(\omega' + \alpha\omega) \geq \mathcal{R}_{\pi_u}(\omega') + \alpha\mathcal{R}_{\pi_u}(\omega) > \mathcal{R}_{\pi_u}(\omega') \text{ every } 0 < \alpha \leq 1,$$

so that $\omega'$ belongs to the closure of $\widehat{P}(\omega')$. Now, if $z \in \widehat{P}(\omega') \cap L_u^+$, then $\mathcal{R}_{\pi_u}(z) > \mathcal{R}_{\pi_u}(\omega') = \overline{p}_u(\omega')$. From $\mathcal{R}_{\pi^u} \leq \overline{p}_u$ on $L_u$, it follows that $\overline{p}_u \cdot z > \overline{p}_u \cdot \omega'$. If we define $X = \{z \in L_u^+ \colon\ \overline{p}_u \cdot z \leq \overline{p}_u \cdot \omega')\}$, the last observation can be rephrased as $X \cap \widehat{P}(\omega') = \emptyset$, so that to each $\omega' \in \Omega$, we can associate a nonzero $\overline{\pi}_{\omega\prime} \in L'$ which separates $\widehat{P}(\omega')$ and $X$, that is,

$$\overline{\pi}_{\omega'} \cdot X \leq \overline{\pi}_{\omega'} \cdot \omega' \leq \overline{\pi}_{\omega'} \cdot \widehat{P}(\omega'). \tag{4.2}$$

Since $\omega' + \omega$ is an interior point of $\widehat{P}(\omega')$, we have $\overline{\pi}_{\omega'} \cdot \omega > 0$, and we can normalize prices letting

$$\overline{p}_u \cdot \omega = \mathcal{R}_{\pi^u}(\omega) = \overline{\pi}_{\omega'} \cdot \omega = 1. \tag{4.3}$$

Let $L_{\omega'}$ be the ordered vector subspace $L_{\omega'} = \bigcup_{\lambda > 0} \lambda[-\omega', \omega']$. Clearly, $L_{\omega'} \subset L_u$. We first claim that $\overline{\pi}_{\omega'} = \overline{p}_u$ on $L_{\omega'}$. Indeed, for every $z \in L_u^+$, we know that

$$\overline{p}_u \cdot z \leq \overline{p}_u \cdot \omega' \implies \overline{\pi}_{\omega'} \cdot z \leq \overline{\pi}_{\omega'} \cdot \omega'.$$

So, using the existence of Lagrange multipliers for a convex programming problem,[9] there exist two real numbers $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ not all equal to zero such that $\lambda_1[\overline{\pi}_{\omega'} \cdot z - \overline{\pi}_{\omega'} \cdot \omega'] \leq \lambda_2[\overline{p}_u \cdot z - \overline{p}_u \cdot \omega']$ for every $z \in L_u^+$. From $\overline{\pi}_{\omega'} \cdot \omega = \overline{p}_u \cdot \omega > 0$, we easily deduce that $(\lambda_1, \lambda_2) \gg 0$. Letting successively $z = 0$ and $z = 2\omega'$, we also see that for some $\lambda > 0$, $\overline{\pi}\omega' \cdot \omega' = \lambda\overline{p}_u \cdot \omega'$ and $\overline{\pi}_{\omega'} \cdot z \leq \overline{p}_u \cdot z$ for every $z \in L_u^+$. The previous inequality holds in particular for every $z \in L_{\omega'}^+$. Since $\omega'$ is an interior point (for the order topology) of $L_{\omega'}^+$, it follows that $\overline{\pi}_{\omega'} = \lambda\overline{p}_u$ on $L_{\omega'}$. Recalling that $\overline{\pi}_{\omega'} \cdot \omega = \overline{p}_u \cdot \omega$, it follows that $\lambda = 1$.

Recalling that for each $i$, we have $\overline{p}_u \cdot \overline{x}_i = \overline{p}_u \cdot \omega_i + \sum_{j \in J} \theta_{ij}\overline{p}_u \cdot \overline{y}_j$, that is,

$$\overline{p}_u \cdot (\overline{x}_i - z'_i) + \sum_{j \in J} \theta_{ij}\overline{p}_u \cdot (z'_j - \overline{y}_j) = \overline{p}_u \cdot (\omega_i - z'_i) + \sum_{j \in J} \theta_{ij}\overline{p}_u \cdot z'_j,$$

we deduce from our first claim:

$$\overline{\pi}_{\omega'} \cdot (\overline{x}_i - z'_i) + \sum_{j \in J} \theta_{ij}\overline{\pi}_{\omega'} \cdot (z'_j - \overline{y}_j) = \overline{\pi}_{\omega'} \cdot (\omega_i - z'_i) + \sum_{j \in J} \theta_{ij}\overline{\pi}_{\omega'} \cdot z'_j,$$

and, thus, for each $i$ and for every $\omega' \in \Omega$,

$$\overline{\pi}_{\omega'} \cdot \overline{x}_i = \overline{\pi}_{\omega'} \cdot \omega_i + \sum_{j \in J} \theta_{ij}\overline{\pi}_{\omega'} \cdot \overline{y}_j\,. \tag{4.4}$$

We next claim that for some $\mu > 0$ (depending on $\omega'$), $\mu\overline{\pi}_{\omega'}$ is a supergradient of $\mathcal{R}_{\pi_u}$ at $\omega'$. Indeed, from $\mathcal{R}_{\pi_u}(z) \geq \mathcal{R}_{\pi_u}(\omega') \implies \overline{\pi}_{\omega'}(z) \geq \overline{\pi}_{\omega'}(\omega')$ for every $z \in L_+$, one deduces, as

[9] A simple proof of the existence of Lagrange multipliers for a convex programming problem can be found in Barbu and Precupanu [18, Chapter 3, Theorem 1.1]. See also [20] and [1, Chapter 5, Theorem 5.77].

previously, the existence of two real numbers $\mu_1 \geq 0$ and $\mu_2 \geq 0$ not all equal to zero such that $\mu_1[\overline{\pi}_{\omega'} \cdot z - \overline{\pi}_{\omega'} \cdot \omega'] \geq \mu_2[\mathcal{R}_{\pi_u}(z) - \mathcal{R}_{\pi_u}(\omega')]$ for every $z \in L+$. As previously, from $\mathcal{R}_{\pi_u}(\omega) = \overline{\pi}_{\omega'} \cdot \omega > 0$, one deduces $(\mu_1, \mu_2) \gg 0$, so that for some $\mu > 0$, $\mu\overline{\pi}_{\omega'} \cdot \omega' = \mathcal{R}_{\pi_u}(\omega')$ and $\mu\overline{\pi}_{\omega'} \cdot z \geq \mathcal{R}_{\pi_u}(z)$ for every $z \in L_+$, which proves the claim.

Fix now $(x, y) \in \prod_{i \in I} P_i^u(\overline{x}_i) \times \prod_{j \in J} Y_j^u$. Let $\omega_0' = \omega - \sum_{i \in I} z_{i0}' + \sum_{j \in J} z_{j0}'$ where for each $i$, $z_{i0}' \in Z_i(\overline{x}_i) \cap L_u$, $z_{i0}' \leq \{0, x_i, \omega_i, \overline{x}_i\}$, and for each $j$, $z_{j0}' \in Z_j(\overline{y}_j) \cap L_u$, $z_{j0}' \geq \{y_j, 0, \overline{y}_j\}$. Once again, this is possible in view of the definition of $L_u$ and our assumptions on $Z_i(\overline{x}_i)$ and $Z_j(\overline{y}_j)$. For every $\omega' \geq \omega'$ in $\Omega$, applying Theorem 2.3 and using (1) and (2) of Proposition 4.1, we get:

$$x_i \in P_i^u(\overline{x}_i) \Longrightarrow \overline{\pi}_{\omega'} \cdot (x_i - z_i') \geq \overline{\pi}_{\omega'} \cdot (\overline{x}_i - z_i') \Longrightarrow \overline{\pi}_{\omega'} \cdot x_i \geq \overline{\pi}_{\omega'} \cdot \overline{x}_i \tag{4.5}$$

and for each $i$, $\pi_i^u(\overline{x}_i - z_i') = \overline{\pi}_{\omega'} \cdot (\overline{x}_i - z_i')$,

$$y_j \in Y_j^u \Longrightarrow \overline{\pi}_{\omega'} \cdot (z_j' - y_j) \geq \overline{\pi}_{\omega'} \cdot (z_j' - \overline{y}_j) \Longrightarrow \overline{\pi}_{\omega'} \cdot y_j \leq \overline{\pi}_{\omega'} \cdot \overline{y}_j \tag{4.6}$$

and for each $j$, $\pi_j^u(z_j' - \overline{y}_j) = \overline{\pi}_{\omega'} \cdot (z_j' - \overline{y}_j)$.

Let $V$ be the $\tau$-neighborhood of 0 referred to in Assumption **A3**. We can assume that $V$ is convex and circled. From (4.2), we deduce that $\overline{\pi}_{\omega'} \cdot V \leq \overline{\pi}_{\omega'} \cdot \omega = 1$, thus that each $\overline{\pi}_{\omega'}$ belongs to $V^0$, the polar set of $V$ in $L'$. Since $L$ is locally convex, it follows from Alaoglu–Bourbaki's theorem that $V^0$ is $\tau$-equicontinuous, thus $\sigma(L', L)$-compact. Passing to a subnet if necessary, we can assume that $\overline{\pi}_{\omega'} \xrightarrow{\sigma(L',L)} \overline{\pi}_u \in L'$ such that $\overline{\pi}_u \cdot \omega = 1$. Passing to limit in the relations (4.5), (4.6) and (4.4), we get $\overline{\pi}_u \cdot x_i \geq \overline{\pi}_u \overline{x}_i$, $\overline{\pi}_u \cdot y_j \leq \overline{\pi}_u \overline{y}_j$, and for each $i$, $\overline{\pi}_u \cdot \overline{x}_i = \overline{\pi}_u \cdot \omega_i + \sum_{j \in J} \theta_{ij} \overline{\pi}_u \cdot \overline{y}_j$, which completes the proof that $(\overline{x}, \overline{y}, \overline{\pi}_u)$ is a quasi-equilibrium of $\mathcal{E}_u$. ■

To go further, we now consider the family $\mathcal{U}$ of all $u > 0$ in $L$ such that the order interval $[-u, u]$ contains $\overline{x}_i, \omega_i, \overline{y}_j$ for all $i \in I$ and $j \in J$ and notice that $\mathcal{U}$ is a directed set. We will apply the previous result to the economies $\mathcal{E}_u$ defined as above and will pass to limit.

The next proposition is the main result of this paper.

**Proposition 4.3.** *Under* **A3***, there exists a price system* $\overline{\pi} \in L'$ *such that* $\overline{\pi} \cdot \omega > 0$ *and* $(\overline{x}, \overline{y}, \overline{\pi})$ *is a quasi-equilibrium of* $\mathcal{E}$. *This quasi-equilibrium is nontrivial if for some* $\lambda > 0$, $\lambda\omega \in \omega + \sum_{j \in J} Y_j - \sum_{i \in I} X_i$.

*Proof.* For each $u \in \mathcal{U}$, in view of the previous proposition and of its proof, let $\overline{\pi}_u \in L'$ such that $\overline{\pi}_u \cdot \omega' \leq \overline{\pi}_u \cdot \widehat{P}(\omega')$ and $(\overline{x}, \overline{y}, \overline{\pi}_u)$ is a quasi-equilibrium of $\mathcal{E}_u$. Let $V$ be the convex and circled $\tau$-neighborhood of 0 such that $\omega' + \omega + V \subseteq \widehat{P}(\omega')$. As previously, $\overline{\pi}_u \cdot V \leq \overline{\pi}_u \cdot \omega = 1$, and passing to a subnet if necessary, we can assume that $\overline{\pi}_u \xrightarrow{\sigma(L',L)} \overline{\pi} \in L'$ such that $\overline{\pi} \cdot \omega = 1$.

We now claim that $\overline{\pi}$ supports the allocation $(\overline{x}, \overline{y})$. To see that, fix now $(x, y) \in \prod_{i \in I} P_i(\overline{x}_i) \times \prod_{j \in J} Y_j$. By construction of $\mathcal{U}$, all $x_i, y_j$ belong to some $L_{u_0}$ for $u_0 \in \mathcal{U}$ and consequently to any $L_u \supset L_{u_0}$. Passing to limit in the relations $\overline{\pi}_u \cdot x_i \geq \overline{\pi}_u \overline{x}_i$ and $\overline{\pi}_u \cdot y_j \leq \overline{\pi}_u \overline{y}_j$, we get $\overline{\pi} \cdot x_i \geq \overline{\pi}\overline{x}_i$ and $\overline{\pi} \cdot y_j \leq \overline{\pi}\overline{y}_j$.

Passing to limit in the relations $\overline{\pi}_u \cdot \overline{x}_i = \overline{\pi}_u \cdot \omega_i + \sum_{j \in J} \theta_{ij} \overline{\pi}_u \cdot \overline{y}_j$, we get that for each $i$, $\overline{\pi} \cdot \overline{x}_i = \overline{\pi} \cdot \omega_i + \sum_{j \in J} \theta_{ij} \overline{\pi} \cdot \overline{y}_j$, proving that $(\overline{x}, \overline{y}, \overline{\pi})$ is a quasi-equilibrium of $\mathcal{E}$.

Finally, assume that for some $\lambda > 0$, $\lambda\omega \in \omega + \sum_{j \in J} Y_j - \sum_{i \in I} X_i$. From $\overline{\pi} \cdot \omega = 1$, we deduce that there exist $x \in \prod_{i \in I} X_i$, $y \in \prod_{j \in J} Y_j$ such that $\overline{\pi} \cdot (\omega + \sum_{j \in J} y_j - \sum_{i \in I} x_i) > 0$. We then have for some $i_0$,

$$\overline{\pi} \cdot x_{i_0} < \overline{\pi} \cdot \omega_{i_0} + \sum_{j \in J} \theta_{i_0 j} \overline{\pi} \cdot y_j \leq \overline{\pi} \cdot \omega_{i_0} + \sum_{j \in J} \theta_{i_0 j} \overline{\pi} \cdot \overline{y}_j = \overline{\pi} \cdot \overline{x}_{i_O},$$

which proves that the quasi-equilibrium is nontrivial. ■

**Remark 4.7.** The condition for nontriviality is in particular satisfied if inaction is possible as well for consumers ($0 \in X_i$) as for producers ($0 \in Y_j$).

Recalling that preferences are non-ordered and that we do not assume any monotonicity property of preferences or free-disposal in production, a consequence of the last proposition is the following theorem which extends any previous equilibrium existence result obtained in the Mas-Colell or Mas-Colell–Richard settings:

**Theorem 4.8.** *Assume **A1, A2, A3** on the commodity space. Then an $\omega$-proper and compact production economy in which inaction is possible as well for consumers as for producers has a nontrivial quasi-equilibrium.*

**Remark 4.9.** As usual, an irreducibility condition on the economy guarantees that the nontrivial equilibrium is actually an equilibrium. A very simple condition, inspired by Arrow–Hahn [16], is the following:

**IR:** For any non-trivial partition $\{I_1, I_2\}$ of the set $I$ of consumers and for any feasible consumption allocation $x$, there exist $x' \in \prod_{i \in I} X_i$ such that
- $x'_i \in P_i(x_i)$ $\forall i \in I_1$;
- $\sum_{i \in I} x'_i \in \omega' + \sum_{j \in J} Y_j$ with, for some $\lambda > 0$, $(\omega' - \omega) \leq \lambda \sum_{i \in I_2} \omega_i$ .

In the next section, we show with examples that this paper is more than a unified proof for old results and allows to obtain the existence of equilibrium in settings of economic interest which are covered by no previous equilibrium existence result.

## 5. Examples

There is a class of examples of commodity spaces which satisfy Assumptions **A1** and **A2** and are not vector lattices. We concentrate here on a few examples of commodity spaces that satisfy also Assumption **A3**.

**Example 5.1.** Consider the vector space

$$L = \left\{ f \in C[0,2] \colon \; f(1) = \tfrac{1}{2}[f(0) + f(2)] \right\}.$$

Clearly, $L$ is a closed vector subspace of the Banach lattice $C[0,2]$, where $C[0,2]$ is equipped with the sup norm $\|f\|_\infty = \sup_{x \in [0,2]} |f(x)|$. The ordered vector space has the following properties.

1. $L$ has order units; for instance constant function $\mathbf{1} \in L$ is an order unit.
2. The positive cone $L_+$ is closed, generating, with a non-empty interior.
3. $L$ is not a vector lattice.
4. $L$ satisfies the Riesz Decomposition Property.[10]
5. The order intervals are norm bounded.

[10] This was shown by I. Namioka [31, p. 45]; see also [32, Example 1.7, p. 14].

In particular, it should be clear from the above properties that assumptions **A1** and **A2** are satisfied.

Moreover, in this case the Decomposition Property guarantees that the Riesz–Kantorovich functionals are additive and their linear extensions are continuous (since the closed unit ball of $L$ is the order interval $[-\mathbf{1}, \mathbf{1}]$). As in Remark 4.6, it is easily deduced that for any $\omega > 0$ and every $m$-tuple $(f_1, \dots, f_m)$ of continuous linear functionals satisfying $f_i(\omega) > 0$ its Riesz–Kantorovich functional is uniformly $\omega$-proper on $L_+$.

**Example 5.2.** Let $L = C^k[0, 1]$, the vector space of all real-valued functions on $[0, 1]$ which are $k$ times continuously differentiable. With the pointwise ordering and the sup norm, $L$ is an ordered topological vector space such that:

1. $L$ is not a vector lattice.
2. $L$ has order units; for instance $\mathbf{1} \in L$ (and hence the positive cone $L_+$ is generating).
3. The positive cone is norm closed.
4. The order intervals are norm bounded.
5. $L$ satisfies the Riesz Decomposition Property. [11]
6. $L'$ is a vector lattice and its lattice operations are given by the Riesz–Kantorovich formulas.

We indicate here how to prove (6) without using (5). One thus has to prove separately the two claims contained in Assertion (6). Actually, both can be deduced from the fact that every positive linear functional on $L$ has a unique extension to a positive linear functional on all of $C[0, 1]$. It follows from this that the norm dual of $L$ is order isomorphic to the norm dual of $C[0, 1]$ and thus, since $L^\sim = ca[0, 1]$ is a Riesz space, $L'$ is likewise a Riesz space. Using the same extension property and the norm density of $L$ in $C[0, 1]$, it is easily proved that the lattice operations of $L'$ satisfy the Riesz–Kantorovich formula.

Clearly, Assumptions **A1** and **A2** are satisfied. Once more, since the Riesz–Kantorovich functionals are additive, **A3** follows from similar arguments to the ones used in Remark 4.6.

**Remark 5.3.** In the two examples above, the $\omega$-uniform properness of the Riesz Kantorovich functional is implied by the facts that $L'$ is a vector lattice and its lattice operations are given by the Riesz–Kantorovich formulas. One could wonder whether these conditions are equivalent to the $\omega$-uniform properness condition. The following example shows that the answer is not.

**Example 5.4.** Let $L = \mathbb{R}^\ell$ be ordered by a closed convex pointed and generating cone $K$. It follows that $K$ has a non-empty interior and that the order intervals of $L$ are compact. In this case, it is well-known that $K$ has the decomposition property if and only if $K$ has exactly $\ell$ extremal rays, i.e., if and only if $K$ is a lattice cone.

Now fix $\omega > 0$ (i.e., $\omega \neq 0$ and $\omega \in K$) and pick $m$ linear functionals $(f_1, \dots, f_m)$ such that $f_i(\omega) > 0$ for each $i$. If $\omega$ is in the interior of $L_+ = K$, then it is easily seen that $\mathcal{R}_f$ is $\omega$-uniformly proper on $L_+$.[12] However, generically on the finite list of linear functionals, their Riesz–Kantorovich is not additive.

This case is not of real interest for equilibrium existence since when $\omega$ is in the interior of the positive cone then the equilibrium of an $\omega$-proper economy can be proved using classical theorems.

[11] This is asserted by several authors (see for instance [24, p. 9] and [25, pp. 18–20]). For the case $k = 1$, one can find a proof in unpublished lecture notes of C.D. Aliprantis.

[12] Indeed, Let $V$ be a $\tau$-neighborhood of 0 such that $\omega + V \subseteq K$ and $f_i(\omega + V) > 0$ for each $i$. Then $\mathcal{R}_f(\omega + V) > 0$, thus $\mathcal{R}_f(\omega' + \omega + V) \geq \mathcal{R}_f(\omega') + \mathcal{R}_f(\omega + V) > \mathcal{R}_f(\omega')$.

When $\omega$ is on the boundary of the positive cone, equilibrium may not exist (see an example in [11]) and the Riesz–Kantorovich functionals may not be $\omega$-proper. An interesting case is the case where $L_+ = K$ is a polyhedral convex cone (with more than $\ell$ extremal rays) and $\omega$ does not belong to the interior of $K$. It should be noted that for any finite list of continuous linear functionals $(f_k)_{k=1}^K$ such that $f_k(\omega) > 0$ for each $k$, $\mathcal{R}_f$ is $\omega$-proper at $\omega$. However, we do not know at this time if these functionals are uniformly proper at any point $\omega' \geq \omega$.

## References


[1] C. D. Aliprantis and K. C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, $2^{\text{nd}}$ Edition, Springer–Verlag, Heidelberg and New York, 1999.

[2] C. D. Aliprantis, D. J. Brown, and Burkinshaw, Edgeworth equilibria, *Econometrica* **55** (1987), 1109–1137.

[3] C. D. Aliprantis, D. J. Brown, and Burkinshaw, Edgeworth equilibria in production economies *J. Econom. Theory* **43** (1987), 252–291.

[4] C. D. Aliprantis, D. J. Brown, I. A. Polyrakis, and J. Werner, Portfolio dominance and optimality in infinite security markets, *J. Math. Econom.* **30** (1998), 347–366.

[5] C. D. Aliprantis, D. J. Brown, and J. Werner, Minimum-cost portfolio insurance, *J. Econom. Dynam. Control* **24** (2000), 1703–1719.

[6] C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, Academic Press, New York and London, 1985.

[7] C. D. Aliprantis and O. Burkinshaw, *Locally Solid Riesz Spaces with Applications to Economics*, American Mathematical Society, Mathematical Surveys and Monographs, Volume 105, 2003.

[8] C. D. Aliprantis, M. Florenzano, F. Martins Da Rocha and R. Tourky, Equilibrium analysis in financial markets with countably many securities, *J. Math. Econom.* **40** (2004), 683–699.

[9] C. D. Aliprantis, M. Florenzano, and R. Tourky, Linear and non-linear price decentralization, *J. Econom. Theory*, forthcoming.

[10] C. D. Aliprantis, M. Florenzano, and R. Tourky, General equilibrium analysis in ordered topological vector spaces, *J. Math. Econom.* **40** (2004), 247–269.

[11] C. D. Aliprantis, P. K. Monteiro, and R. Tourky, Non-marketed options, non-existence of equilibria, and non-linear prices, *J. Econom. Theory*, **114** (2004), 345–357.

[12] C. D. Aliprantis and R. Tourky, The super order dual of an ordered vector space and the Riesz–Kantorovich formula, *Trans.. Amer. Math. Soc.* **354** (2002), 2055–2077.

[13] C. D. Aliprantis, R. Tourky, and N. C. Yannelis, The Riesz–Kantorovich formula and general equilibrium theory, *J. Math. Econom.* **34** (2000), 55–76.

[14] C. D. Aliprantis, R. Tourky, and N. C. Yannelis, A theory of value: equilibrium analysis beyond vector lattices, *J. Econom. Theory* **100** (2001), 22–72.

[15] N. Allouch and M. Florenzano, Edgeworth and Walras equilibria of an arbitrage-free exchange economy, *Econom. Theory*, **23** (2004), 353–370.

[16] K. J. Arrow and F. H. Hahn, *General Competitive Analysis*, Holden-Day, San Francisco, 1971.

[17] J. P. Aubin, *Mathematical Methods of Game and Economic Theory*, North–Holland, Amsterdam and New York, 1979.

[18] V. Barbu and T. Precupanu, *Convexity and Optimization in Banach spaces*, D. Reidel Publishing Company, Boston and Lancaster, 1986.

[19] G. Debreu and H. Scarf, A limit theorem on the core of an economy, *Internat. Econom. Rev.* **4** (1963), 235–246.

[20] K. Fan, I. L. Glicksberg, and A. J. Hoffman, Systems of inequalities involving convex functions, *Proc. Amer. Math. Soc.* **13** (1957), 617–622.

[21] M. Florenzano, Edgeworth equilibria, fuzzy core and equilibria of a production economy without ordered preferences, *J. Math. Anal. Appl.* **153** (1990), 18–36.

[22] M. Florenzano, *General Equilibrium Analysis: Existence and Optimality Properties of Equilibria,* Kluwer Academic Publishers, Boston and London, 2003.

[23] M. Florenzano and V. Marakulin, Production equilibria in vector lattices, *Econom. Theory* **17** (2001), 577–598.

[24] L. Fuchs, Riesz groups, *Ann. Scuola Norm. Sup. Pisa* **19** (1965), 1–34.

[25] L. Fuchs *Riesz Vector Spaces and Riesz Algebras,* Queen's papers in pure and applied mathematics, No 1, Queen's University, Kingston, Ontario, Canada, 1966.

[26] P. J. Laurent, *Approximation et Optimisation*, Herman, Paris, 1972.

[27] A. Mas-Colell, The price equilibrium existence problem in topological vector lattices, *Econometrica* **54** (1986), 1039–1055.

[28] A. Mas-Colell, Valuation equilibria and Pareto optimum revisited, in W. Hildenbrand and A. Mas-Colell (eds.), *Contributions to Mathematical Economics. In honor of Gérard Debreu*, North-Holland, Amsterdam, 317–331, 1986.

[29] A. Mas-Colell and S. F. Richard, A new approach to the existence of equilibria in vector lattices, *J. Econom. Theory* **53** (1991), 1–11.

[30] J. J. Moreau, *Fonctionnelles convexes*, Séminaire sur les équations aux dérivées partielles, Collège de France, 1966–1967.

[31] I. Namioka, *Partially Ordered Linear Topological Spaces*, Mem. Amer. Math. Soc., **24**, Providence, RI, 1957.

[32] A. L. Peressini, *Ordered Topological Vector Spaces*, Harper & Row, New York and London, 1967.

[33] K. Podczeck, Equilibria in vector lattices without ordered preferences or uniform properness, *J. Math. Econom.* **25** (1996), 465–485.

[34] S. F. Richard, A new approach to production equilibria in vector lattices, *J. Math. Econom.* **18** (1989), 41–56.

[35] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*, Springer–Verlag, Heildelberg and New York, 1998.

[36] H. H. Schaefer, *Topological Vector Spaces*, Springer Verlag, New York and Heildelberg, 1971.

[37] R. Tourky, A new approach to the limit theorem on the core of an economy in vector lattices, *J. Econom. Theory* **78** (1998), 321–328.

[38] R. Tourky, The limit theorem on the core of a production economy in vector lattices with unordered preferences, *Econom. Theory* **14** (1999), 219–226.

[39] W. R. Zame, Competitive equilibria in production economies with an infinite dimensional space, *Econometrica*, **55** (1987), 1075–1108.