Gerard Debreu on the Integration of Correspondences*

M. Ali Khan,[†] Kali P. Rath[‡] and Yeneng Sun[§]

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Abstract

In 1967, Gerard Debreu presented an integral of a correspondence by taking a higher-dimensional point of view whereby a correspondence is seen as a function in a Banach space and thereby integrated as a consequence of the theory of Bochner integration. In this survey paper, we locate Debreu's paper from the viewpoint of subsequent work, both in the theory of embeddings of convex-valued correspondences as functions in a Banach space and the application of Bochner and Gelf'and integration in mathematical economics. We also follow up Debreu's suggested parallelism between two lines of development, the "statistical one and the economic one," by presenting the authors' recent results on integration of correspondences with respect to a vector measure.

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[†]Department of Economics, The Johns Hopkins University, Baltimore, MD 21218.

[‡]Department of Economics and Econometrics, University of Notre Dame, Notre Dame, IN 46556.

[§]Institute for Mathematical Sciences, National University of Singapore, 3 Prince George's Park, Singapore 118402; also Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543.

1 Introduction

In their monograph ([69, p.1], Segal-Kunze write

> [I]ntegration is significant, not only as a vital tool in analysis and as the culmination of the calculus, but also as an intrinsically beautiful and complete theory, in which elements of geometry and algebra, as well as analysis, are merged. Even so, [a student's] understanding of the subject will proceed more rapidly if he has some definite, if general, knowledge of what sort of thing it is and how it is related to the subjects he is already familiar with and if he sees why it has aroused such interest.

Segal-Kunze are focussed on the integral of a function rather than a correspondence and they do not have mathematical economics in mind. However, their observations surely have relevance to the theory of integration of correspondences, and particularly as it has been motivated by, and found application in, mathematical economics: both general competitive analysis and game theory. It is thus of particular interest that Gerard Debreu does not include his work on the integration of correspondences, the so-called Debreu integral (see ([66]) for example), in his collection [30] of *Twenty Papers in Mathematical Economics.* Perhaps this is an instance of what Paul Samuelson [65, p.1266] calls the "serendipity of irrelevant issues".

> As so often happens in the history of scientific thought, one gains attention and appreciation for one's innovations for reasons that prove to have been irrelevancies.

In this context, Debreu's introduction to his 1965 paper [28] repays a careful consideration. His very first paragraph relies on the phrases "proper mathematical formulation" and "natural context."

> The traditional economic concept of a set of agents, each of which cannot influence the outcome of their collective activity but certain coalitions of which can influence that outcome, has received its proper mathematical formulation by means of measure theory. [M]easure theory indeed appears as the natural context in which to study economic competition.

A correspondence is introduced as a "standard operation in the analysis of economic equilibrium" that associates with every element a from a finite set of agents A, a nonempty subset $\phi(a)$ of a "real Banach space S (the commodity space)," and the sum of the sets $\phi(a)$ over the agents is defined as

$$\sum_{a\in A}\phi(a) = \{z \in S : z = \sum_{a\in A} f(a), f : A \to S \text{ such that for all } a \in A, f(a) \in \phi(a)\}.$$

This formal definition of the sum of sets is worthy of notice because it sets the stage for Debreu's stance to the 1965 Aumann paper [14] on the integration of set-valued functions. Taking this "fundamental contribution" as his point of departure, he emphasized "several directions" from which his article could be seen as an extension of this work. In the first of these directions, the natural/artificial dichotomy remains in play.

> The first extension aims at replacing [the] assumption that the set of agents is an analytic set by the assumption that it is a measurable space. From the viewpoint of economic interpretation, this generalization is important, for the identification of economic agents with points of an analytic set seems artificial, unlike the assumption that every countable union of coalitions is a coalition, and that the complement of every coalition is a coalition.

It is interesting that in [13], Aumann also characterizes a market model with an atomless measure space of agents as a "most natural model," and that in [3, 4, 5], Armstrong-Richter introduce their work by an insistence of the artificiality of the countably additive assumption. Since this is not our primary concern in this essay, we move to the second direction one in which convenience rather than importance is emphasized.

> The second extension consists of introducing three criteria for the measurability of a correspondence in addition to the criterion used by Aumann, the four criteria being essentially equivalent. Since in the various situations encountered in the theory of integration of correspondences, one of these criteria is often far easier to apply than the others, this four-fold diversity is of great convenience.

This work receives a consolidation in the 1977 monograph Castaing-Valadier [21].

> The third extension attempts to relax the assumption of finite dimensionality of the space S.

This extension anticipates the subsequent work [16], [49, 52], [55, 56], [53], [70, 71].

What is of particular interest to the project pursued in this essay is Debreu's meticulous care in recognizing the work done by statisticians. He writes

> Extensive work has been done in statistics ... on mathematical problems closely related to those with which this article deals. [N]o attempt will be made to compare in detail the mathematical results obtained in the two lines of development, the statistical one and the economic one. It may be noted that among the propositions required by the economic theory appear generalizations of several results of the statistical theory. This paper represents a laying up of many strands ... and it could not have achieved its present form without the conversations I had with economists, mathematicians and statisticians over the last year.

While our primary aim in this essay is to palcement of Debreu's contribution to the theory of integration, we shall also attempt to show the cost of this neglect in inter-disciplinary communication for the subject of mathematical economics.

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