Econ 204 2010

Lecture 1

Outline

1. Introductions
2. About the Course and Other Administrative Details
3. Methods of Proof
4. Equivalence Relations
5. Cardinality

Introductions

Welcome

- 204
- Berkeley Economics
- UC Berkeley
- Berkeley
- California

- US...

Introductions

- Chris Shannon
- Ivan Balbuzanov
- Oleksa Shvets

About the Course

- **Schedule:** Lectures MTWThF 9:00 - 11:00 here (180 Tan), often going over so don't schedule anything before 11:30

  Sections: MTWThF 1 - 2:30 and 2:30 - 4:00, in 608-7 Evans (please try to split up evenly)

  Office hours: Chris Shannon MTWThF 11:00 - 12:00 here or 511 Evans, also by appt.

  Oleksa + Ivan MTWThF 3:00 - 5:00 608-1 Evans (switching off days)

- **Final Exam:** Wednesday August 18, 9:00 am - 12:00 pm, 180 Tan Hall

- **Prerequisites:** Math 53-54 at Berkeley or equivalent
  - 4 semesters college mathematics
  - linear algebra
  - multivariable calculus
  - rigorous approach - theorems stated carefully and some proofs given
  - stream for engineers and scientists

Course requirements:

- problems sets: 6 total

  (**no late problem sets...no exceptions**)

- exam

- reading/working on your own

Grade: 20% problem sets (5 highest scores out of 6), 80% final exam

Grading in First Year Economics Courses:

- median grade = B+ : solid command of material

- A and A- are very good grades, A+ for truly exceptional work

- B : ready to go on to further work...a B in 204 means you are ready to go on to 201a/b, 202a/b, 240a/b

- B- : very marginal, but we won't make you take the class again. B- in 204 means you will have a very hard time in 201a/b. Recommend you take Math 53 and 54 this year, maybe Math 104, come back next year to retake 204 and take 201a/b. B- is a passing grade, but you must maintain a B average

- C: not passing. Definitely not ready for 201a/b, 202a/b, 240a/b. Take Math 53-54 this year, maybe Math 104, retake 204 next year

- 204 with at least a B- (or a waiver from 204 requirement) is a strictly enforced prerequisite for enrollment in 201a/b

- F: means you didn't take the final exam. Be sure to withdraw if you don't or can't take the final.

Resources:

Book: de la Fuente, *Mathematical Methods and Models for Economists*

Lecture notes: for every lecture + supplements for several topics

Be sure to read Corrections Handout with dlF

Seek out other references

This class is not normal...

- lectures
- expectations
- classroom stuff

Goals for 204

- reduce heterogeneity of math backgrounds for students in Econ graduate classes

- advance everyone's math skills and knowledge

- present some particular concepts and results used in first-year economics courses 201a/b, 202a/b, 240a/b

- challenge everyone - so not everyone will understand everything

- develop basic math skills and knowledge needed to work as a professional economist and read academic economics

- develop ability to read and evaluate purported proofs...essential for reading and working in all branches of economics - theoretical, empirical, experimental

- develop ability to compose simple proofs...essential to working in all branches of economics - theoretical, empirical, experimental

- cover selected material from real analysis and linear algebra at moderate level of abstraction (considerably more advanced and abstract than Math 53 + 54)

- **not** to review Math 53 + 54. If you are weak on this material, take Math 53-54 this year, and take 204 next year.

Learning by Doing

- to learn this sort of mathematics you need to do more than just read the book and notes and listen to lectures

- active reading: work through each line, be sure you know how to get from one line to the next

- active listening: follow each step as we work through arguments in class

- working problems: the most valuable part of the class

- working in groups strongly encouraged...
- but, always try to work through all of the problems before talking to others
- everyone must write up his/her own solutions
- best test of understanding: can you explain it to others

Methods of Proof

What is a proof? The million dollar question...

Main Methods of Proof:

- deduction
- contraposition
- induction
- contradiction

We'll examine each of these in turn.

Proof by Deduction

Proof by Deduction: A list of statements, the last of which is the statement to be proven. Each statement in the list is either

- an axiom: a fundamental assumption about mathematics, or part of definition of the object under study; or
- a previously established theorem; or
- follows from previous statements in the list by a valid rule of inference

Proof by Deduction

Example: Prove that the function $f(x) = x^2$ is continuous at $x = 5$.

Recall from one-variable calculus that $f(x) = x^2$ is continuous at $x = 5$ means

$$\forall \varepsilon > 0 \; \exists \delta > 0 \text{ s.t. } |x - 5| < \delta \Rightarrow |f(x) - f(5)| < \varepsilon$$

That is, "for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever x is within δ of 5, $f(x)$ is within ε of $f(5)$."

To prove the claim, we must systematically verify that this definition is satisfied.

Proof. Let $\varepsilon > 0$ be given. Let

$$\delta = \min\left\{1, \frac{\varepsilon}{11}\right\} > 0$$

Where did that come from ? Suppose $|x-5| < \delta$. Since $\delta \leq 1$, $4 < x < 6$, so $9 < x+5 < 11$ and $|x+5| < 11$. Then

$$\begin{aligned}
|f(x) - f(5)| &= |x^2 - 25| \\
&= |(x+5)(x-5)| \\
&= |x+5||x-5| \\
&< 11 \cdot \delta \\
&\leq 11 \cdot \frac{\varepsilon}{11} \\
&= \varepsilon
\end{aligned}$$

Thus, we have shown that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|x-5| < \delta \Rightarrow |f(x) - f(5)| < \varepsilon$, so f is continuous at $x = 5$. $\square$

Proof by Contraposition

not P

Recall some basics of logic. P, Q

$\neg P$ means "P is false."

and $P \wedge Q$ means "P is true *and* Q is true."

or $P \vee Q$ means "P is true *or* Q is true (or possibly both)."

$\neg P \wedge Q$ means $(\neg P) \wedge Q$; $\neg P \vee Q$ means $(\neg P) \vee Q$.

$P \Rightarrow Q$ means "whenever P is satisfied, Q is also satisfied."

Formally, $P \Rightarrow Q$ is equivalent to $\neg P \vee Q$.

P is false, or P is true, so Q is true

Proof by Contraposition

The *contrapositive* of the statement $P \Rightarrow Q$ is the statement $\neg Q \Rightarrow \neg P$.

Theorem 1. *$P \Rightarrow Q$ is true if and only if $\neg Q \Rightarrow \neg P$ is true.*

Proof. Suppose $P \Rightarrow Q$ is true. Then either P is false, or Q is true (or possibly both). Therefore, either $\neg P$ is true, or $\neg Q$ is false (or possibly both), so $\neg(\neg Q) \vee (\neg P)$ is true, that is, $\neg Q \Rightarrow \neg P$ is true.

Q or $\neg P$

Conversely, suppose $\neg Q \Rightarrow \neg P$ is true. Then either $\neg Q$ is false, or $\neg P$ is true (or possibly both), so either Q is true, or P is false (or possibly both), so $\neg P \vee Q$ is true, so $P \Rightarrow Q$ is true. $\square$

Claim: $P(n)$ $\forall n \in \mathbb{N}$

Prove Base case n_0

Proof by Induction

Induction assp: $P(n)$ true for $n \in \mathbb{N}$

Show: $P(n+1)$ true

We illustrate with an example:

Theorem 2. *For every* $n \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

i.e. $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

Proof. **Base step** $n = 0$: LHS $= \sum_{k=1}^{0} k =$ the empty sum $=$ 0. RHS $= \frac{0 \cdot 1}{2} = 0$

So the claim is true for $n = 0$.

Induction step: Suppose

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \text{ for some } n > 0$$

We must show that

$$\sum_{k=1}^{n+1} k = \frac{(n+1)((n+1)+1)}{2}$$

$$
\begin{aligned}
\text{LHS} &= \sum_{k=1}^{n+1} k \\
&= \sum_{k=1}^{n} k + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) \text{ by the Induction hypothesis} \\
&= (n+1)\left(\frac{n}{2}+1\right) \\
&= \frac{(n+1)(n+2)}{2} \\
\text{RHS} &= \frac{(n+1)((n+1)+1)}{2} \\
&= \frac{(n+1)(n+2)}{2} = \text{LHS}
\end{aligned}
$$

So by mathematical induction, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for all $n \in \mathbf{N}_0$. $\square$

Proof by Contradiction

Assume the negation of what is claimed, and work toward a contradiction.

Theorem 3. *There is no rational number q such that $q^2 = 2$.*

= rational numbers

Proof. Suppose $q^2 = 2$ where $q \in \mathbf{Q}$. Then we can write $q = \frac{m}{n}$ for some integers $m, n \in \mathbf{Z}$. Moreover, we can assume that m and n have no common factor; if they did, we could divide it out.

integers

$$2 = q^2 = \frac{m^2}{n^2}$$

Therefore, $m^2 = 2n^2$, so m^2 is even.

We claim that m is even. If not, then m is odd, so $m = 2p + 1$ for some $p \in \mathbf{Z}$. Then

$$\begin{aligned} m^2 &= (2p+1)^2 \\ &= 4p^2 + 4p + 1 \\ &= 2(2p^2 + 2p) + 1 \end{aligned}$$

which is odd, contradiction. Therefore, m is even, so $m = 2r$ for some $r \in \mathbf{Z}$.

$$\begin{aligned} 4r^2 &= (2r)^2 \\ &= m^2 \\ &= 2n^2 \\ n^2 &= 2r^2 \end{aligned}$$

So n^2 is even, which implies (by the argument given above) that n is even. Therefore, $n = 2s$ for some $s \in \mathbf{Z}$, so m and n have a common factor, namely 2, contradiction. Therefore, there is no rational number q such that $q^2 = 2$. $\square$

Equivalence Relations

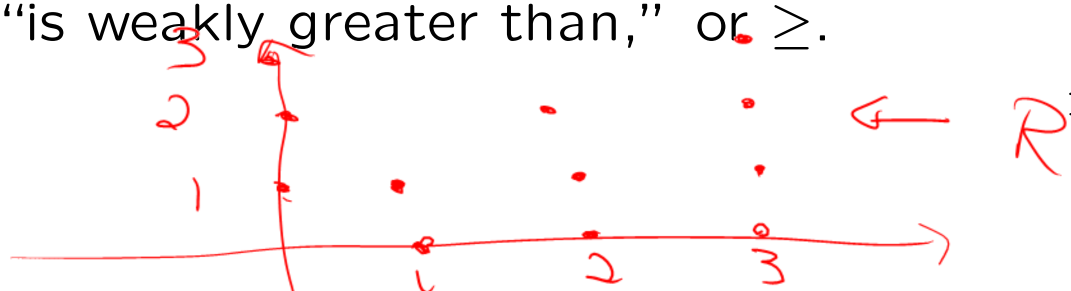
Definition 1. *A* binary relation R *from* X *to* Y *is a subset* $R \subseteq X \times Y$. *We write* xRy *if* $(x, y) \in R$ *and "not* xRy*" if* $(x, y) \notin R$. $R \subseteq X \times X$ *is a* binary relation on X.

Example: Suppose $f : X \to Y$ is a function from X to Y. The binary relation $R \subseteq X \times Y$ defined by

$$xRy \iff f(x) = y$$

is exactly the graph of the function f. A function can be considered a binary relation R from X to Y such that for each $x \in X$ there exists exactly one $y \in Y$ such that $(x, y) \in R$.

Example: Suppose $X = \{1, 2, 3\}$ and R is the binary relation on X given by $R = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3)\}$. This is the binary relation "is weakly greater than," or $\geq$.



Equivalence Relations

Definition 2. *A binary relation* R *on* X *is*

(i) reflexive *if* $\forall x \in X, xRx$

(ii) symmetric *if* $\forall x, y \in X, xRy \Leftrightarrow yRx$

(iii) transitive *if* $\forall x, y, z \in X, (xRy \wedge yRz) \Rightarrow xRz$

Definition 3. *A binary relation* R *on* X *is an* equivalence relation *if it is reflexive, symmetric and transitive.*

Equivalence Relations

Definition 4. *Given an equivalence relation R on X, write*

$$[x] = \{y \in X : xRy\}$$

$[x]$ *is called the* equivalence class containing x.

The set of equivalence classes is the quotient *of X with respect to R, denoted X/R.* "X mod R"

real numbers

Example: The binary relation $\geq$ on $\mathbf{R}$ is not an equivalence relation because it is not symmetric.

Example: Let $X = \{a, b, c, d\}$ and

$$R = \{(a,a), (a,b), (b,a), (b,b), (c,c), (c,d), (d,c), (d,d)\}$$

R is an equivalence relation (why?) and the equivalence classes of R are $\{a, b\}$ and $\{c, d\}$. $X/R = \{\{a, b\}, \{c, d\}\}$

Equivalence Relations

The equivalence classes of an equivalence relation form a *partition* of X: every element of X belongs to exactly one equivalence class.

Theorem 4. *Let R be an equivalence relation on X. Then $\forall x \in X, x \in [x]$. Given $x, y \in X$, either $[x] = [y]$ or $[x] \cap [y] = \emptyset$.*

Proof. If $x \in X$, then xRx because R is reflexive, so $x \in [x]$.

Suppose $x, y \in X$. If $[x] \cap [y] = \emptyset$, we're done. So suppose $[x] \cap [y] \neq \emptyset$. We must show that $[x] = [y]$, i.e. that the elements of $[x]$ are exactly the same as the elements of $[y]$.

Choose $z \in [x] \cap [y]$. Then $z \in [x]$, so xRz. By symmetry, zRx. Also $z \in [y]$, so yRz. By symmetry again, zRy. Now choose $w \in [x]$. By definition, xRw. Since zRx and R is transitive, zRw. By symmetry, wRz. Since zRy, wRy by transitivity again. By symmetry, yRw, so $w \in [y]$, which shows that $[x] \subseteq [y]$.
Similarly, $[y] \subseteq [x]$, so $[x] = [y]$. $\square$

Cardinality

Definition 5. *Two sets A, B are* numerically equivalent *(or* have the same cardinality*) if there is a bijection $f : A \to B$, that is, a function $f : A \to B$ that is 1-1 ($a \neq a' \Rightarrow f(a) \neq f(a')$), and onto ($\forall b \in B\ \exists a \in A$ s.t. $f(a) = b$).*

Example: $A = \{2, 4, 6, \ldots, 50\}$ is numerically equivalent to the set $\{1, 2, \ldots, 25\}$ under the function $f(n) = 2n$.

$B = \{1, 4, 9, 16, 25, 36, 49 \ldots\} = \{n^2 : n \in \mathbf{N}\}$ is numerically equivalent to $\mathbf{N}$.

Cardinality

A set is either finite or infinite. A set is *finite* if it is numerically equivalent to $\{1, \ldots, n\}$ for some n. A set that is not finite is *infinite*.

In particular, $A = \{2, 4, 6, \ldots, 50\}$ is finite, $B = \{1, 4, 9, 16, 25, 36, 49 \ldots\}$ is infinite.

A set is *countable* if it is numerically equivalent to the set of natural numbers $\mathrm{N} = \{1, 2, 3, \ldots\}$. A set that is not countable is called *uncountable*.

Cardinality

Example: The set of integers $\mathbf{Z}$ is countable.

$$\mathbf{Z} = \{0, 1, -1, 2, -2, \ldots\}$$

Define $f : \mathbf{N} \to \mathbf{Z}$ by

$$\begin{aligned} f(1) &= 0 \\ f(2) &= 1 \\ f(3) &= -1 \\ &\vdots \\ f(n) &= (-1)^n \left\lfloor \frac{n}{2} \right\rfloor \end{aligned}$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. It is straightforward to verify that f is one-to-one and onto.

Cardinality

Theorem 5. *The set of rational numbers* $\mathbf{Q}$ *is countable.*

"Picture Proof":

$$\begin{aligned} \mathbf{Q} &= \left\{\frac{m}{n} : m, n \in \mathbf{Z}, n \neq 0\right\} \\ &= \left\{\frac{m}{n} : m \in \mathbf{Z}, n \in \mathbf{N}\right\} \end{aligned}$$

f(4) - -

f(1) f(2)

f(3)

n \ m	0	1	-1	2	-2
1	0	1	-1	2	-2
2	$\cancel{0}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\cancel{1}$	-1
3	$\cancel{0}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{2}{3}$
4	$\cancel{0}$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$
5	$\cancel{0}$	$\frac{1}{5}$	$-\frac{1}{5}$	$\frac{2}{5}$	$-\frac{2}{5}$

Go back and forth on upward-sloping diagonals, omitting the repeats:

$$
\begin{aligned}
f(1) &= 0 \\
f(2) &= 1 \\
f(3) &= \frac{1}{2} \\
f(4) &= -1 \\
&\vdots
\end{aligned}
$$

$f : \mathbf{N} \rightarrow \mathbf{Q}$, f is one-to-one and onto.