Econ 204 2010

Lecture 5

Outline

1. Properties of Real Functions (Sect. 2.6, cont.)
2. Monotonic Functions
3. Cauchy Sequences and Complete Metric Spaces
4. Contraction Mappings
5. Contraction Mapping Theorem

Announcements:

- PS1 due today here & now
- PS2 available today due Tues 8/3 in lecture
- revised notes for today lecture 5 posted last night
Properties of Real Functions

Here we first study properties of functions from $\mathbb{R}$ to $\mathbb{R}$, making use of the additional structure we have in $\mathbb{R}$ as opposed to general metric spaces.

Let $f : X \to \mathbb{R}$ where $X \subseteq \mathbb{R}$. We say $f$ is bounded above if

$$f(X) = \{ r \in \mathbb{R} : f(x) = r \text{ for some } x \in X \}$$

is bounded above. Similarly, we say $f$ is bounded below if $f(X)$ is bounded below. Finally, $f$ is bounded if $f$ is both bounded above and bounded below, that is, if $f(X)$ is a bounded set.
Extreme Value Theorem

Theorem 1 (Thm. 6.23, Extreme Value Theorem). Let $a, b \in \mathbb{R}$ with $a \leq b$ and let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then $f$ assumes its minimum and maximum on $[a, b]$. That is, if

$$M = \sup_{t \in [a, b]} f(t) \quad m = \inf_{t \in [a, b]} f(t)$$

then $\exists t_M, t_m \in [a, b]$ such that $f(t_M) = M$ and $f(t_m) = m$.

Proof. Let

$$M = \sup\{f(t) : t \in [a, b]\}$$

If $M$ is finite, then for each $n$, we may choose $t_n \in [a, b]$ such that $M \geq f(t_n) \geq M - \frac{1}{n}$ (if we couldn’t make such a choice, then $M - \frac{1}{n}$ would be an upper bound and $M$ would not be the
supremum). If $M$ is infinite, choose $t_n$ such that $f(t_n) \geq n$. By the Bolzano-Weierstrass Theorem, $\{t_n\}$ contains a convergent subsequence $\{t_{n_k}\}$, with

$$\lim_{k \to \infty} t_{n_k} = t_0 \in [a, b]$$

Since $f$ is continuous,

$$f(t_0) = \lim_{t \to t_0} f(t) = \lim_{k \to \infty} f(t_{n_k}) = M$$

so $M$ is finite and

$$f(t_0) = M = \sup\{f(t) : t \in [a, b]\}$$

so $f$ attains its maximum and is bounded above.

The argument for the minimum is similar.
Intermediate Value Theorem Redux

**Theorem 2** (Thm. 6.24, Intermediate Value Theorem). Suppose \( f : [a, b] \rightarrow \mathbb{R} \) is continuous, and \( f(a) < d < f(b) \). Then there exists \( c \in (a, b) \) such that \( f(c) = d \).

**Proof.** Let

\[
B = \{ t \in [a, b] : f(t) < d \}
\]
a \in B, so \( B \neq \emptyset \). By the Supremum Property, \( \sup B \) exists and is real so let \( c = \sup B \). Since \( a \in B, c \geq a \). \( B \subseteq [a, b] \), so \( c \leq b \). Therefore, \( c \in [a, b] \). We claim that \( f(c) = d \).

Let

\[
t_n = \min \left\{ c + \frac{1}{n}, b \right\} \geq c
\]
Either $t_n > c$, in which case $t_n \not\in B$, or $t_n = c$, in which case $t_n = b$ so $f(t_n) > d$, so again $t_n \not\in B$; in either case, $f(t_n) \geq d$. Since $f$ is continuous at $c$, $f(c) = \lim_{n \to \infty} f(t_n) \geq d$ (Theorem 3.5 in de la Fuente).

Since $c = \sup B$, we may find $s_n \in B$ such that

$$c \geq s_n \geq c - \frac{1}{n}$$

Since $s_n \in B$, $f(s_n) < d$. Since $f$ is continuous at $c$, $f(c) = \lim_{n \to \infty} f(s_n) \leq d$ (Theorem 3.5 in de la Fuente).

Since $d \leq f(c) \leq d$, $f(c) = d$. Since $f(a) < d$ and $f(b) > d$, $a \neq c \neq b$, so $c \in (a, b)$.
Monotonic Functions

Definition 1. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is monotonically increasing if

$$y > x \Rightarrow f(y) \geq f(x)$$

Monotonic functions are very well-behaved...
Monotonic Functions

**Theorem 3** (Thm. 6.27). Let \( a, b \in \mathbb{R} \) with \( a < b \), and let \( f : (a, b) \to \mathbb{R} \) be monotonically increasing. Then the one-sided limits

\[
\begin{align*}
\text{right-hand limit } f(t^+) &= \lim_{u \to t^+} f(u) = \lim_{n \to \infty} f(t_n) \quad \text{for } t_n \to t^+, t_n > t \\
\text{left-hand limit } f(t^-) &= \lim_{u \to t^-} f(u) = \lim_{n \to \infty} f(s_n) \quad \text{for } s_n \to t^-, s_n < t
\end{align*}
\]

exist and are real numbers for all \( t \in (a, b) \).

**Proof.** This is analogous to the proof that a bounded monotone sequence converges. \( \square \)
Monotonic Functions

We say that $f$ has a *simple jump discontinuity* at $t$ if the one-sided limits $f(t^-)$ and $f(t^+)$ both exist but $f$ is not continuous at $t$.

Note that there are two ways $f$ can have a simple jump discontinuity at $t$: either $f(t^+) \neq f(t^-)$, or $f(t^+) = f(t^-) \neq f(t)$.

The previous theorem says that monotonic functions have **only** simple jump discontinuities. Note that monotonicity also implies that $f(t^-) \leq f(t) \leq f(t^+)$. So a monotonic function has a discontinuity at $t$ if and only if $f(t^+) \neq f(t^-)$.
Monotonic Functions

A monotonic function is continuous “almost everywhere” — except for at most countably many points.

**Theorem 4** (Thm. 6.28). Let $a, b \in \mathbb{R}$ with $a < b$, and let $f : (a, b) \to \mathbb{R}$ be monotonically increasing. Then

$$D = \{ t \in (a, b) : f \text{ is discontinuous at } t \}$$

is finite (possibly empty) or countable.

**Proof.** If $t \in D$, then $f(t^-) < f(t^+)$ (if the left- and right-hand limits agreed, then by monotonicity they would have to equal $f(t)$, so $f$ would be continuous at $t$). $\mathbb{Q}$ is dense in $\mathbb{R}$, that is, if
any subset of a countable set is either empty, finite, or countable.

x, y ∈ ℝ and x < y then ∃r ∈ ℚ such that x < r < y. So for every
t ∈ D we may choose r(t) ∈ ℚ such that

\[ f(t^-) < r(t) < f(t^+) \]

This defines a function \( r : D \to ℚ \). Notice that

\[ s > t \Rightarrow f(s^-) \geq f(t^+) \]

so

\[ s > t, s, t ∈ D \Rightarrow r(s) > f(s^-) \geq f(t^+) > r(t) \]

so \( r(s) ≠ r(t) \). Therefore, \( r \) is one-to-one, so it is a bijection
from \( D \) to a subset of \( ℚ \). Thus \( D \) is finite or countable.  

\[ r : D \to ℚ \quad 1-1 \]

\[ r : D \to r(D) \subseteq ℚ \quad \text{bijection} \]
Cauchy Sequences and Complete Metric Spaces

Roughly, a metric space is complete if “every sequence that ought to converge to a limit has a limit to converge to.”

Recall that $x_n \to x$ means

$$\forall \varepsilon > 0 \; \exists N(\varepsilon/2) \text{ s.t. } n > N(\varepsilon/2) \Rightarrow d(x_n, x) < \frac{\varepsilon}{2}$$

Observe that if $n, m > N(\varepsilon/2)$, then

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
Cauchy Sequences and Complete Metric Spaces

This motivates the following definition:

**Definition 2.** A sequence \( \{x_n\} \) in a metric space \((X, d)\) is Cauchy if

\[
\forall \varepsilon > 0 \; \exists N(\varepsilon) \; s.t. \; n, m > N(\varepsilon) \implies d(x_n, x_m) < \varepsilon
\]

A Cauchy sequence is trying really hard to converge, but there may not be anything for it to converge to.
Cauchy Sequences and Complete Metric Spaces

Any sequence that \textbf{does} converge must be Cauchy:

**Theorem 5** (Thm. 7.2). \textit{Every convergent sequence in a metric space is Cauchy.}

\textit{Proof}. We just did it: Let \( x_n \to x \). For every \( \varepsilon > 0 \) \( \exists N \) such that \( n > N \Rightarrow d(x_n, x) < \varepsilon/2 \). Then

\[
m, n > N \Rightarrow d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]
Example: Let $X = (0, 1]$ and $d$ be the Euclidean metric. Let $x_n = \frac{1}{n}$. Then $x_n \to 0$ in $E^1$, so $\{x_n\}$ is Cauchy in $E^1$. Thus $\{x_n\}$ is Cauchy in $(X, d)$. But $\{x_n\}$ does not converge in $(X, d)$.

The Cauchy property depends only on the sequence and the metric $d$, not on the ambient metric space:

$\{x_n\}$ is Cauchy in $(X, d)$, but $\{x_n\}$ does not converge in $(X, d)$ because the point it is trying to converge to $(0)$ is not an element of $X$. 
Complete Metric Spaces and Banach Spaces

Where does every Cauchy sequence get what it wants?

**Definition 3.** A *metric space* \((X, d)\) is **complete** if every Cauchy sequence \(\{x_n\} \subseteq X\) converges to a limit \(x \in X\).

**Definition 4.** A Banach space is a *normed space* that is complete in the metric generated by its norm.
Complete Metric Spaces and Banach Spaces

Example: Consider the earlier example of $X = (0, 1]$ with $d$ the usual Euclidean metric. The sequence $\{x_n\}$ with $x_n = \frac{1}{n}$ is Cauchy but does not converge, so $((0, 1], d)$ is not complete.

Example: $\mathbb{Q}$ is not complete in the Euclidean metric. To see this, let

$$x_n = \frac{\lfloor 10^n \sqrt{2} \rfloor}{10^n}$$

where $\lfloor y \rfloor$ is the greatest integer less than or equal to $y$; $x_n$ is just equal to the decimal expansion of $\sqrt{2}$ to $n$ digits past the decimal point. Clearly, $x_n$ is rational. $|x_n - \sqrt{2}| \leq 10^{-n}$, so $x_n \to \sqrt{2}$ in $\mathbb{E}^1$, so $\{x_n\}$ is Cauchy in $\mathbb{E}^1$, hence Cauchy in $\mathbb{Q}$; since $\sqrt{2} \notin \mathbb{Q}$, $\{x_n\}$ is not convergent in $\mathbb{Q}$, so $\mathbb{Q}$ is not complete.
Complete Metric Spaces and Banach Spaces

**Theorem 6** (Thm. 7.10). \( \mathbb{R} \) is complete with the usual metric (so \( \mathbb{E}^1 \) is a Banach space).

**Proof.** Suppose \( \{x_n\} \) is a Cauchy sequence in \( \mathbb{R} \). Fix \( \varepsilon > 0 \). Find \( N(\varepsilon/2) \) such that

\[
n, m > N(\varepsilon/2) \Rightarrow |x_n - x_m| < \frac{\varepsilon}{2}
\]

Let

\[
\alpha_n = \sup\{x_k : k \geq n\}
\]
\[
\beta_n = \inf\{x_k : k \geq n\}
\]

Fix \( m > N(\varepsilon/2) \). Then

\[
k \geq m \Rightarrow k > N(\varepsilon/2) \Rightarrow x_k < x_m + \frac{\varepsilon}{2}
\]

\[
\Rightarrow \alpha_m = \sup\{x_k : k \geq m\} \leq x_m + \frac{\varepsilon}{2}
\]
Since $\alpha_m < \infty$,

$$\limsup_{n \to \infty} x_n = \lim_{n \to \infty} \alpha_n \leq \alpha_m \leq x_m + \frac{\varepsilon}{2}$$

since the sequence $\{\alpha_n\}$ is decreasing. Similarly,

$$\liminf_{n \to \infty} x_n \geq x_m - \frac{\varepsilon}{2}$$

Therefore,

$$0 \leq \limsup_{n \to \infty} x_n - \liminf_{n \to \infty} x_n \leq \varepsilon$$

Since $\varepsilon$ is arbitrary,

$$\limsup_{n \to \infty} x_n = \liminf_{n \to \infty} x_n \in \mathbb{R}$$

Thus $\lim_{n \to \infty} x_n$ exists and is real, so $\{x_n\}$ is convergent. \qed
Complete Metric Spaces and Banach Spaces

**Theorem 7** (Thm. 7.11). $\mathbb{E}^n$ is complete for every $n \in \mathbb{N}$.

*Proof.* See de la Fuente. \qed
Complete Metric Spaces and Banach Spaces

**Theorem 8** (Thm. 7.9). Suppose $(X, d)$ is a complete metric space and $Y \subseteq X$. Then $(Y, d) = (Y, d|_Y)$ is complete if and only if $Y$ is a closed subset of $X$.

**Proof.** Suppose $(Y, d)$ is complete. We need to show that $Y$ is closed. Consider a sequence $\{y_n\} \subseteq Y$ such that $y_n \to_{(X, d)} x \in X$. Then $\{y_n\}$ is Cauchy in $X$, hence Cauchy in $Y$; since $Y$ is complete, $y_n \to_{(Y, d)} y$ for some $y \in Y$. Therefore, $y_n \to_{(X, d)} y$.

By uniqueness of limits, $y = x$, so $x \in Y$. Thus $Y$ is closed.

Conversely, suppose $Y$ is closed. We need to show that $Y$ is complete. Let $\{y_n\}$ be a Cauchy sequence in $Y$. Then $\{y_n\}$ is Cauchy in $X$, hence convergent, so $y_n \to_{(X, d)} x$ for some $x \in X$. Since $Y$ is closed, $x \in Y$, so $y_n \to_{(Y, d)} x \in Y$. Thus $Y$ is complete. \qed
Complete Metric Spaces and Banach Spaces

**Theorem 9** (Thm. 7.12). Given $X \subseteq \mathbb{R}^n$, let $C(X)$ be the set of bounded continuous functions from $X$ to $\mathbb{R}$ with

$$
\|f\|_{\infty} = \sup\{|f(x)| : x \in X\}
$$

Then $C(X)$ is a Banach space.
Constructions

**Definition 5.** Let \((X, d)\) be a nonempty complete metric space. An operator is a function \(T : X \to X\).

An operator \(T\) is a contraction of modulus \(\beta\) if \(\beta < 1\) and

\[
d(T(x), T(y)) \leq \beta d(x, y) \quad \forall x, y \in X
\]

A contraction shrinks distances by a **uniform** factor \(\beta < 1\).
Contractions

**Theorem 10.** *Every contraction is uniformly continuous.*

*Proof.* Fix $\varepsilon > 0$. Let $\delta = \frac{\varepsilon}{\beta}$. Then $\forall x, y$ such that $d(x, y) < \delta$,

$$d(T(x), T(y)) \leq \beta d(x, y) < \beta \delta = \varepsilon$$

Note that a contraction is Lipschitz continuous with Lipschitz constant $\beta < 1$ (and hence also uniformly continuous).
Contractions and Fixed Points

**Definition 6.** A fixed point of an operator $T$ is point $x^* \in X$ such that $T(x^*) = x^*$. 
\[ x^* = T(x^*) \]
Contraction Mapping Theorem

**Theorem 11** (Thm. 7.16, Contraction Mapping Theorem). Let \((X, d)\) be a nonempty complete metric space and \(T : X \rightarrow X\) a contraction with modulus \(\beta < 1\). Then

1. \(T\) has a unique fixed point \(x^*\).

2. For every \(x_0 \in X\), the sequence \(\{x_n\}\) where

\[
x_1 = T(x_0), x_2 = T(x_1) = T(T(x_0)), \ldots, x_n = T(x_{n-1})\text{ for each } n
\]

converges to \(x^*\).
Note that the theorem asserts both the existence and uniqueness of the fixed point, as well as giving an algorithm to find the fixed point of a contraction.

Also note that the algorithm generates a sequence that converges to the fixed point for any initial point $x_0$.

Later in the course we will discuss more general fixed point theorems which, in contrast, only guarantee existence, and are not constructive.
Proof. Define the sequence \( \{x_n\} \) as above by first fixing \( x_0 \in X \) and then letting \( x_n = T(x_{n-1}) = T^n(x_0) \) for \( n = 1, 2, \ldots \), where \( T^n = T \circ T \circ \ldots \circ T \) is the \( n \)-fold iteration of \( T \). We first show that \( \{x_n\} \) is Cauchy, and hence converges to a limit \( x \). Then

\[
\begin{align*}
    d(x_{n+1}, x_n) &= d(T(x_n), T(x_{n-1})) \\
    &\leq \beta d(x_n, x_{n-1}) = \beta d(T(x_{n-1}), T(x_{n-2})) \\
    &\leq \beta^2 d(x_{n-1}, x_{n-2}) \\
    &\vdots \\
    &\leq \beta^n d(x_1, x_0)
\end{align*}
\]
Then for any \( n > m \),
\[
d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\
\leq \left( \beta^{n-1} + \beta^{n-2} + \cdots + \beta^m \right) d(x_1, x_0) \\
= d(x_1, x_0) \sum_{\ell=m}^{n-1} \beta^\ell \\
< d(x_1, x_0) \sum_{\ell=m}^{\infty} \beta^\ell \\
= \frac{\beta^m}{1 - \beta} d(x_1, x_0) \quad \text{(sum of a geometric series)}
\]

Fix \( \varepsilon > 0 \). Since \( \frac{\beta^m}{1 - \beta} \to 0 \) as \( m \to \infty \), choose \( N(\varepsilon) \) such that for any \( m > N(\varepsilon) \), \( \frac{\beta^m}{1 - \beta} < \frac{\varepsilon}{d(x_1, x_0)} \). Then for \( n, m > N(\varepsilon) \),
\[
d(x_n, x_m) \leq \frac{\beta^m}{1 - \beta} d(x_1, x_0) < \varepsilon
\]
Therefore, \( \{x_n\} \) is Cauchy. Since \((X, d)\) is complete, \(x_n \to x^*\) for some \(x^* \in X\).

Next, we show that \(x^*\) is a fixed point of \(T\).

\[
T(x^*) = T \left( \lim_{n \to \infty} x_n \right) \\
= \lim_{n \to \infty} T(x_n) \quad \text{since } T \text{ is continuous} \\
= \lim_{n \to \infty} x_{n+1} = x^*
\]

so \(x^*\) is a fixed point of \(T\).

Finally, we show that there is at most one fixed point. Suppose \(x^*\) and \(y^*\) are both fixed points of \(T\), so \(T(x^*) = x^*\) and \(T(y^*) = y^*\).
Then

\[ d(x^*, y^*) = d(T(x^*), T(y^*)) \leq \beta d(x^*, y^*) \]

\[ \Rightarrow (1 - \beta) d(x^*, y^*) \leq 0 \]

\[ \Rightarrow d(x^*, y^*) \leq 0 \]

So \( d(x^*, y^*) = 0 \), which implies \( x^* = y^* \). \qed
Continuous Dependence on Parameters

Theorem 12. (Thm. 7.18', Continuous Dependence on Parameters) Let \((X, d)\) and \((\Omega, \rho)\) be two metric spaces and \(T : X \times \Omega \rightarrow X\). For each \(\omega \in \Omega\) let \(T_\omega : X \rightarrow X\) be defined by

\[
T_\omega(x) = T(x, \omega)
\]

Suppose \((X, d)\) is complete, \(T\) is continuous in \(\omega\), that is \(T(x, \cdot) : \Omega \rightarrow X\) is continuous for each \(x \in X\), and \(\exists \beta < 1\) such that \(T_\omega\) is a contraction of modulus \(\beta\) \(\forall \omega \in \Omega\). Then the fixed point function \(x^* : \Omega \rightarrow X\) defined by

\[
x^*(\omega) = T_\omega(x^*(\omega))
\]

is continuous.
Blackwell’s Sufficient Conditions

An important result due to Blackwell gives a set of sufficient conditions for an operator to be a contraction that is particularly useful in dynamic programming problems.

Let $X$ be a set, and let $B(X)$ be the set of all bounded functions from $X$ to $\mathbb{R}$. Then $(B(X), \| \cdot \|_\infty)$ is a normed vector space.

Notice that below we use shorthand notation that identifies a constant function with its constant value in $\mathbb{R}$, that is, we write interchangeably $a \in \mathbb{R}$ and $a : X \to \mathbb{R}$ to denote the function such that $a(x) = a \ \forall x \in X$. 
Blackwell’s Sufficient Conditions

Theorem 13. (Blackwell’s Sufficient Conditions) Consider $B(X)$ with the sup norm $\| \cdot \|_\infty$. Let $T : B(X) \rightarrow B(X)$ be an operator satisfying

1. (monotonicity) $f(x) \leq g(x) \ \forall x \in X \Rightarrow (Tf)(x) \leq (Tg)(x) \ \forall x \in X$

2. (discounting) $\exists \beta \in (0, 1)$ such that for every $a \geq 0$ and $x \in X$,

$$(T(f + a))(x) \leq (Tf)(x) + \beta a$$

Then $T$ is a contraction with modulus $\beta$. 
Proof. Fix \( f, g \in B(X) \). By the definition of the sup norm, 
\[
f(x) \leq g(x) + \|f - g\|_\infty \quad \forall x \in X
\]
Then 
\[
(Tf)(x) \leq (T(g + \|f - g\|_\infty))(x) \quad \forall x \in X \quad \text{(monotonicity)}
\]
\[
\leq (Tg)(x) + \beta \|f - g\|_\infty \quad \forall x \in X \quad \text{(discounting)}
\]
Thus 
\[
(Tf)(x) - (Tg)(x) \leq \beta \|f - g\|_\infty \quad \forall x \in X
\]
Reversing the roles of \( f \) and \( g \) above gives 
\[
(Tg)(x) - (Tf)(x) \leq \beta \|f - g\|_\infty \quad \forall x \in X
\]
Thus 
\[
\|T(f) - T(g)\|_\infty \leq \beta \|f - g\|_\infty
\]
Thus \( T \) is a contraction with modulus \( \beta \) \qed