Announcements

- PS 2 due tomorrow (8/3) in lecture
- revised notes for today (lecture 6) posted last night ≈ 6pm

# Econ 204 2010

## Lecture 6

### Outline

1. Open Covers
2. Compactness
3. Sequential Compactness
4. Totally Bounded Sets
5. Heine-Borel Theorem
6. Extreme Value Theorem

# Open Covers

**Definition 1.** *A collection of sets*

$$\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$$

*in a metric space* $(X, d)$ *is an* open cover *of* $A$ *if* $U_\lambda$ *is open for all* $\lambda \in \Lambda$ *and*

$$\cup_{\lambda \in \Lambda} U_\lambda \supseteq A$$

Notice that $\Lambda$ may be finite, countably infinite, or uncountable.

# Compactness

**Definition 2.** *A set $A$ in a metric space is* compact *if every open cover of $A$ contains a finite subcover of $A$. In other words, if $\{U_\lambda : \lambda \in \Lambda\}$ is an open cover of $A$, there exist $n \in \mathbf{N}$ and $\lambda_1, \cdots, \lambda_n \in \Lambda$ such that*

$$A \subseteq U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}$$

This definition does **not** say "$A$ has a finite open cover" (fortunately, since this is vacuous...).

Instead for **any** arbitrary open cover you must specify a finite subcover of this **given** open cover.

# Compactness

**Example:** $(0,1]$ is not compact in $\mathbf{E}^1$.

To see this, let

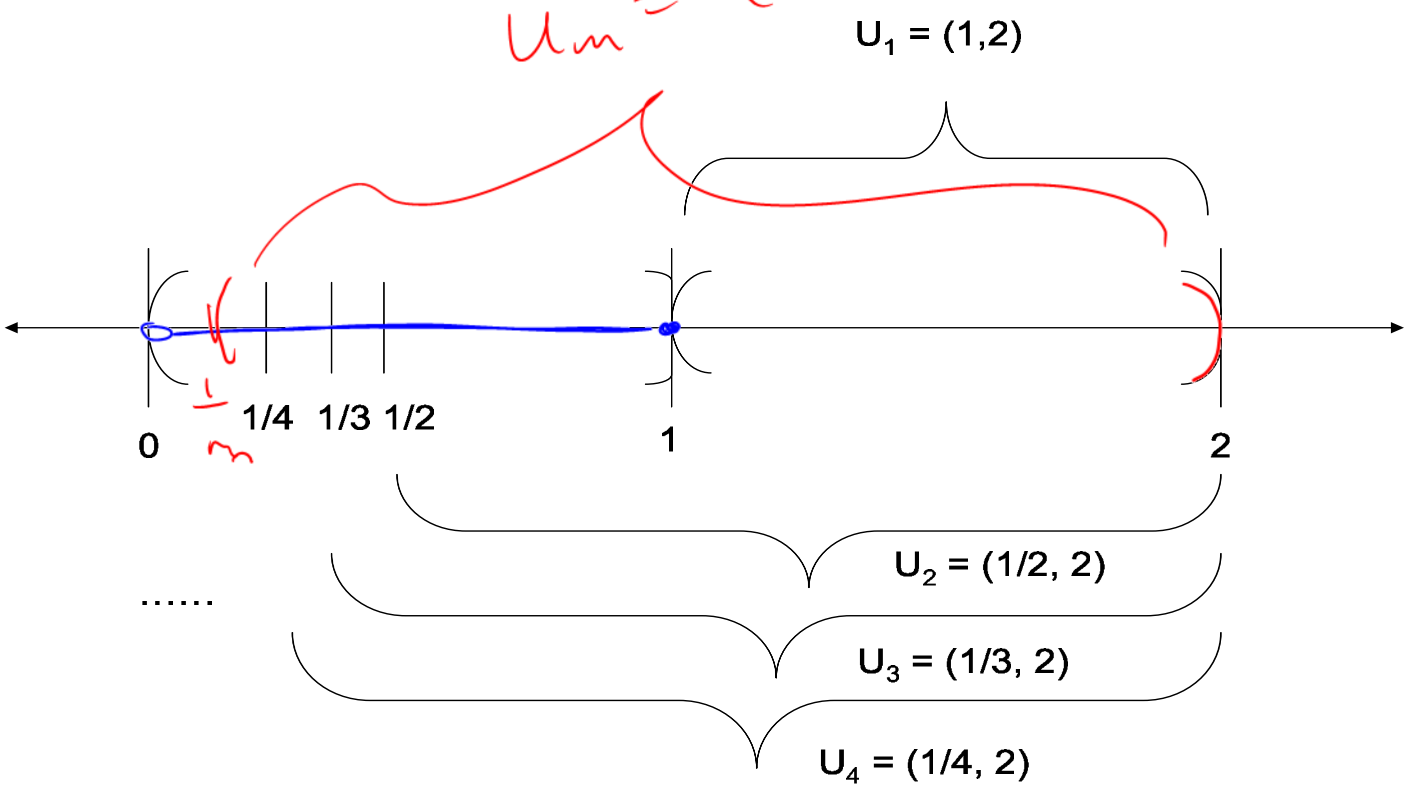
$$\mathcal{U} = \left\{ U_m = \left( \frac{1}{m}, 2 \right) : m \in \mathbf{N} \right\}$$

Then

$$\cup_{m \in \mathbf{N}} U_m = (0,2) \supset (0,1]$$

$A = (0, 1]$

$U_m = (\frac{1}{m}, 2)$




$$\bigcup_m (\frac{1}{m}, 2) = (0, 2) \supseteq (0, 1]$$

Given any finite subset $\{U_{m_1}, \ldots, U_{m_n}\}$ of $\mathcal{U}$, let

$$m = \max\{m_1, \ldots, m_n\}$$

Then

$$\cup_{i=1}^{n} U_{m_i} = U_m = \left(\frac{1}{m}, 2\right) \not\supseteq (0, 1]$$

So $(0, 1]$ is not compact.

What about $[0, 1]$? This argument doesn't work...

# Compactness

**Example:** $[0, \infty)$ is closed but not compact. in $E^1$

To see that $[0, \infty)$ is not compact, let

$$\mathcal{U} = \{U_m = (-1, m) : m \in \mathbf{N}\}$$

$[0, \infty) \subseteq \bigcup_m (-1, m) = (-1, \infty)$

Given any finite subset

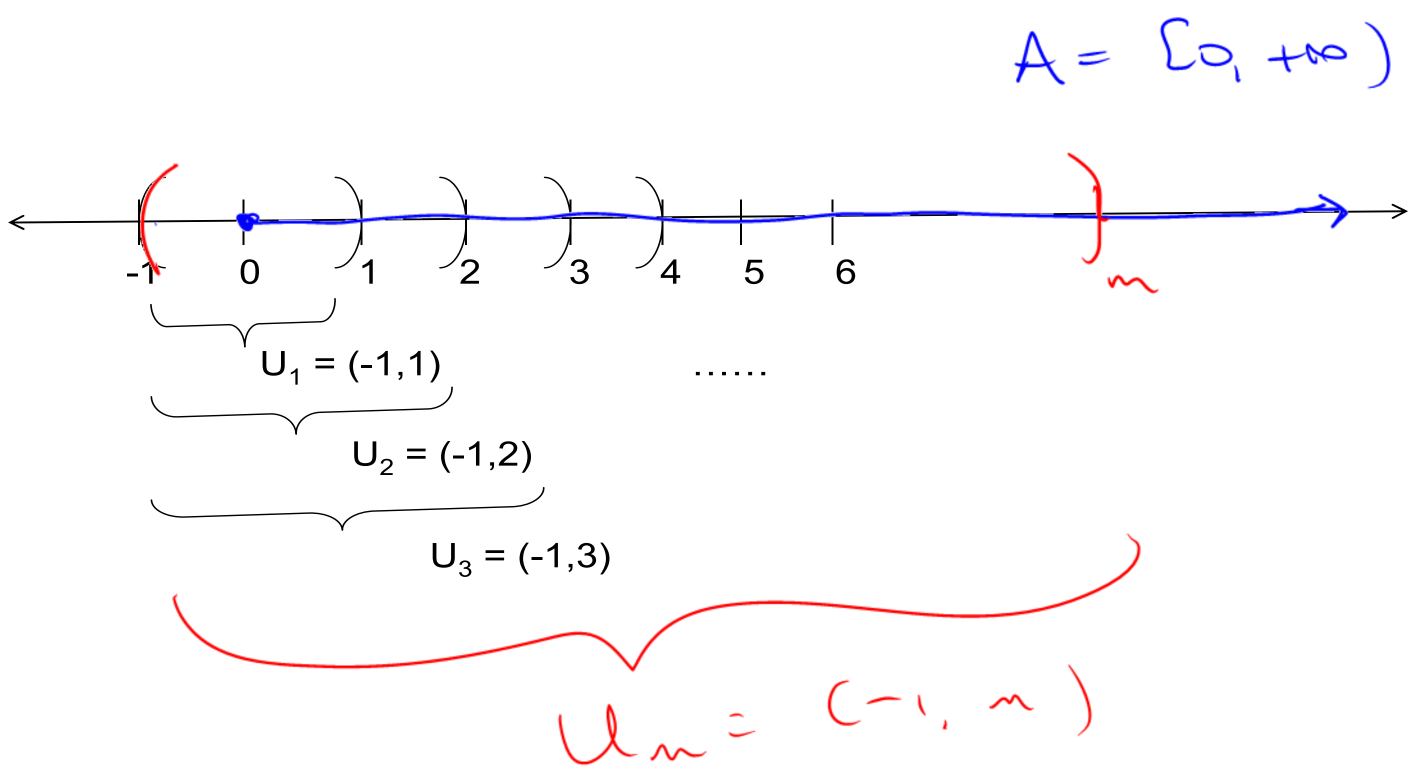
$$\{U_{m_1}, \ldots, U_{m_n}\}$$

of $\mathcal{U}$, let

$$m = \max\{m_1, \ldots, m_n\}$$

Then

$$U_{m_1} \cup \cdots \cup U_{m_n} = (-1, m) \not\supseteq [0, \infty)$$


A = [0, +∞)
-1
0
1
2
3
4
5
6
m
U1 = (-1,1)
......
U2 = (-1,2)
U3 = (-1,3)
Um = (-1, m)

# Compactness

**Theorem 1** (Thm. 8.14)**.** *Every closed subset $A$ of a compact metric space $(X, d)$ is compact.*

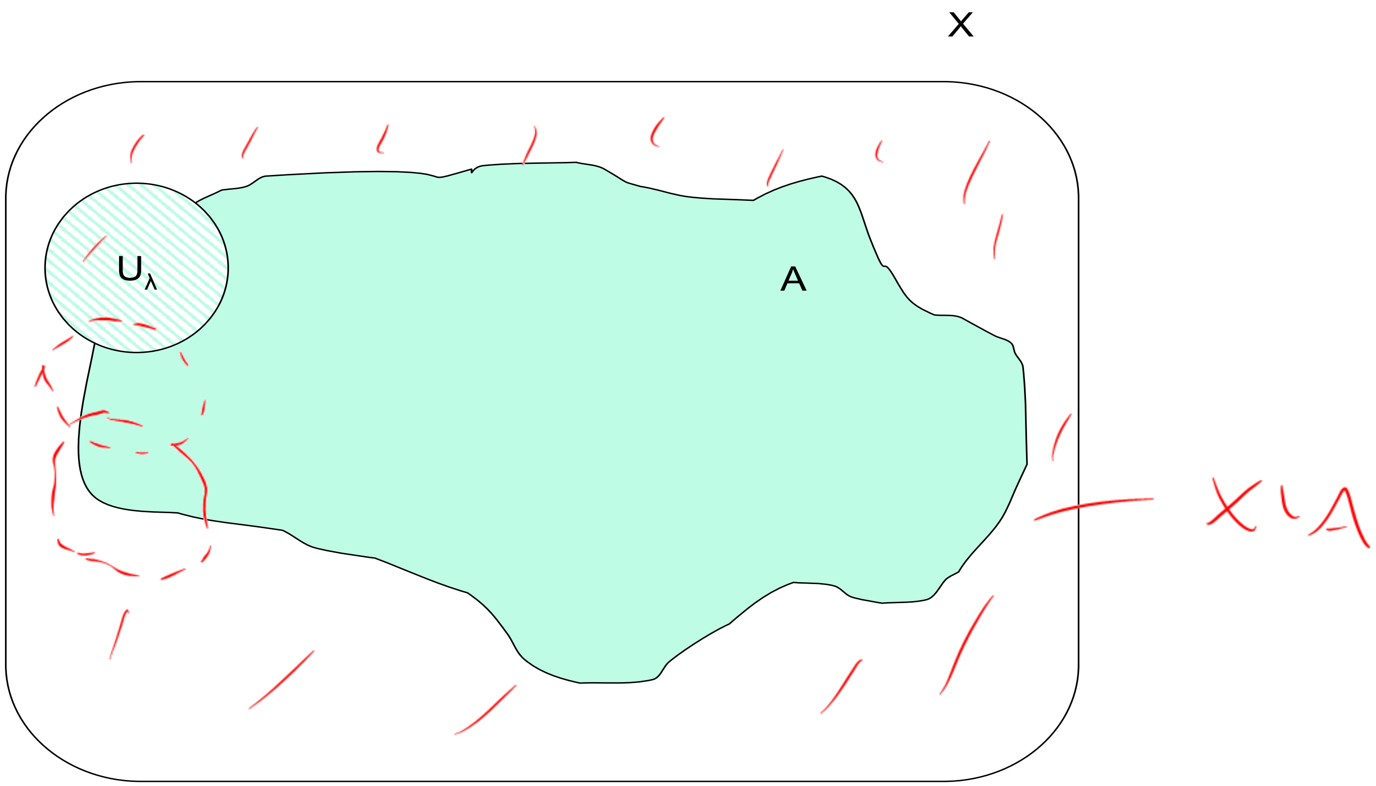
*Proof.* Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover of $A$. In order to use the compactness of $X$, we need to produce an open cover of $X$. There are two ways to do this:

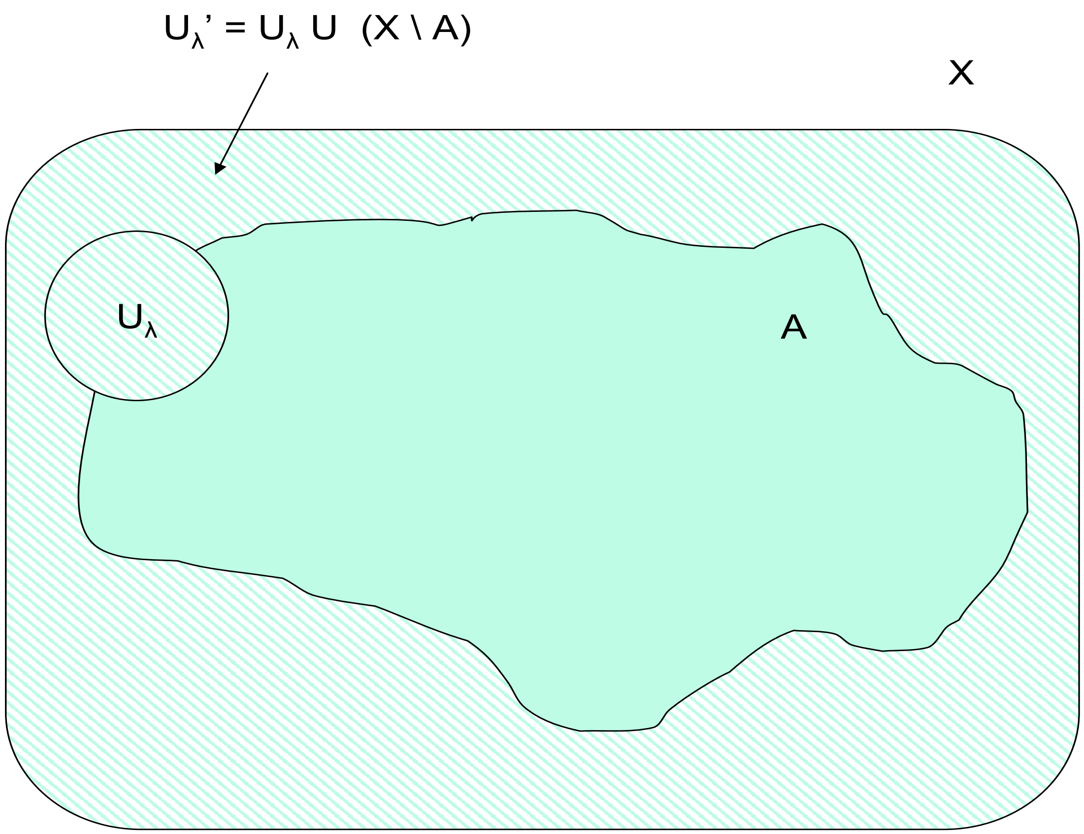
open → A closed ⇒ X \ A open

$$\begin{aligned} U'_\lambda &= U_\lambda \cup (X \setminus A) \\ \Lambda' &= \Lambda \cup \{\lambda_0\},\ U_{\lambda_0} = X \setminus A \end{aligned}$$

We choose the first path, and let

$$U'_\lambda = U_\lambda \cup (X \setminus A)$$


X
U
λ
A
X \ A


Uλ' = Uλ U (X \ A)
X
Uλ
A

Since $A$ is closed, $X \setminus A$ is open; since $U_\lambda$ is open, so is $U'_\lambda$.

Then $x \in X \Rightarrow x \in A$ or $x \in X \setminus A$. If $x \in A$, $\exists \lambda \in \Lambda$ s.t. $x \in U_\lambda \subseteq U'_\lambda$. If instead $x \in X \setminus A$, then $\forall \lambda \in \Lambda$, $x \in U'_\lambda$. Therefore, $X \subseteq \cup_{\lambda \in \Lambda} U'_\lambda$, so $\{U'_\lambda : \lambda \in \Lambda\}$ is an open cover of $X$.

Since $X$ is compact,

$$\exists \lambda_1, \ldots, \lambda_n \in \Lambda \text{ s.t. } X \subseteq U'_{\lambda_1} \cup \cdots \cup U'_{\lambda_n}$$

Then

$$\begin{aligned} a \in A &\Rightarrow a \in X \\ &\Rightarrow a \in U'_{\lambda_i} \text{ for some } i \\ &\Rightarrow a \in U_{\lambda_i} \cup (X \setminus A) \\ &\Rightarrow a \in U_{\lambda_i} \end{aligned}$$

so

$$A \subseteq U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}$$

Thus $A$ is compact. $\square$

# Compactness

closed $\not\Rightarrow$ compact, but the converse is true:

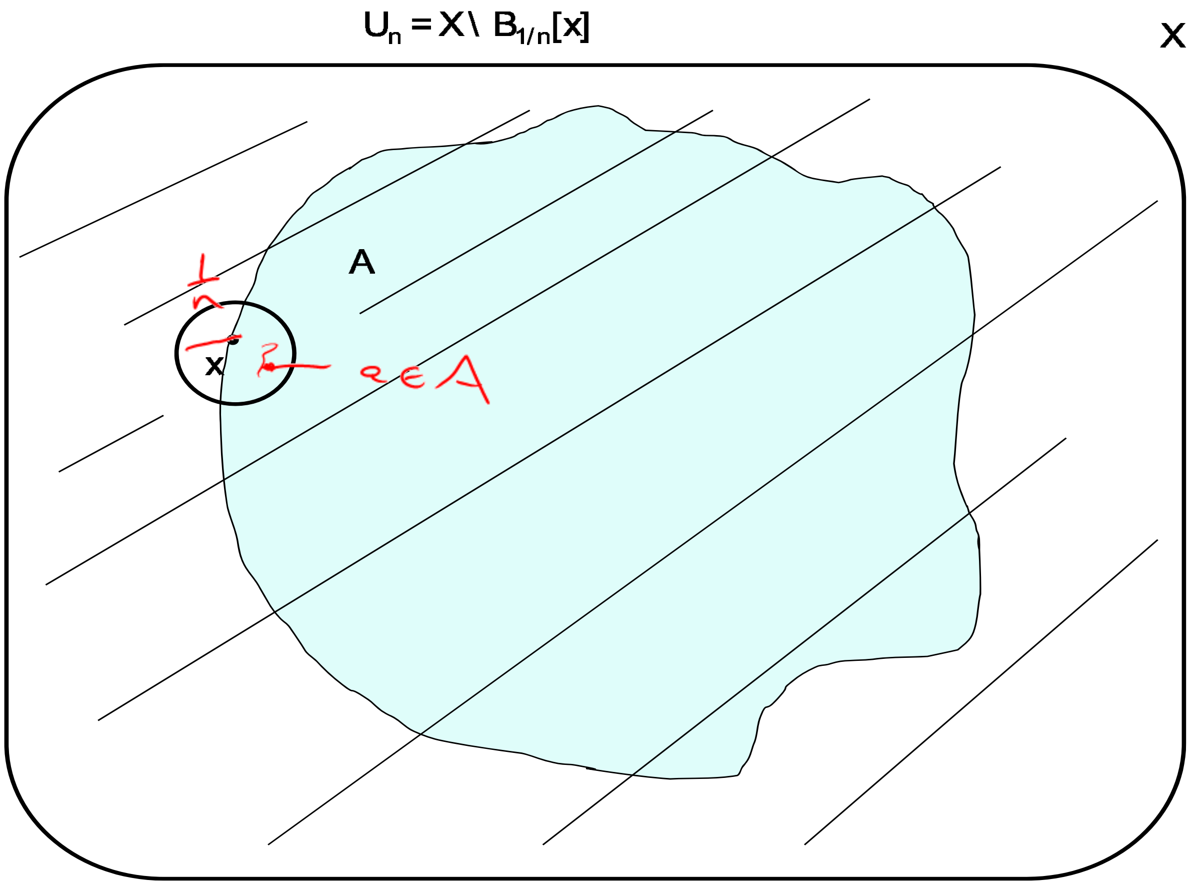
**Theorem 2** (Thm. 8.15). *If $A$ is a compact subset of the metric space $(X, d)$, then $A$ is closed.*

*Proof.* Suppose by way of contradiction that $A$ is not closed. Then $X \setminus A$ is not open, so we can find a point $x \in X \setminus A$ such that, for every $\varepsilon > 0$, $A \cap B_\varepsilon(x) \neq \emptyset$, and hence $A \cap B_\varepsilon[x] \neq \emptyset$. For $n \in \mathbf{N}$, let

$$U_n = X \setminus B_{\frac{1}{n}}[x]$$

open

$\forall \varepsilon > 0 \quad B_\varepsilon(x) \cap A \neq \emptyset$



Each $U_n$ is open, and

$$\cup_{n\in\mathbf{N}} U_n = X \setminus \{x\} \supseteq A \qquad (x \notin A)$$

since $x \notin A$. Therefore, $\{U_n : n \in \mathbf{N}\}$ is an open cover for $A$. Since $A$ is compact, there is a finite subcover $\{U_{n_1}, \ldots, U_{n_k}\}$. Let $n = \max\{n_1, \ldots, n_k\}$. Then

$$\begin{aligned} U_n &= X \setminus B_{\frac{1}{n}}[x] \\ &\supseteq X \setminus B_{\frac{1}{n_j}}[x] \ (j = 1, \ldots, k) \\ U_n &\supseteq \cup_{j=1}^{k} U_{n_j} \\ &\supseteq A \end{aligned}$$

But $A \cap B_{\frac{1}{n}}[x] \neq \emptyset$, so $A \not\subseteq X \setminus B_{\frac{1}{n}}[x] = U_n$, a contradiction which proves that $A$ is closed. $\square$

# Sequential Compactness

**Definition 3.** *A set* $A$ *in a metric space* $(X, d)$ *is* sequentially compact *if every sequence of elements of* $A$ *contains a convergent subsequence whose limit lies in* $A$*.*

# Sequential Compactness

**Theorem 3** (Thms. 8.5, 8.11)**.** *A set $A$ in a metric space $(X, d)$ is compact if and only if it is sequentially compact.*

*Proof.* Suppose $A$ is compact. We will show that $A$ is sequentially compact.

If not, we can find a sequence $\{x_n\}$ of elements of $A$ such that no subsequence converges to **any** element of $A$. Recall that $a$ is a cluster point of the sequence $\{x_n\}$ means that

$$\forall \varepsilon > 0 \quad \{n : x_n \in B_\varepsilon(a)\} \text{ is infinite}$$

and this is equivalent to the statement that there is a subsequence $\{x_{n_k}\}$ converging to $a$. Thus, **no** element $a \in A$ can be a cluster point for $\{x_n\}$, and hence

$$\forall a \in A \;\; \exists \varepsilon_a > 0 \text{ s.t. } \{n : x_n \in B_{\varepsilon_a}(a)\} \text{ is finite} \tag{1}$$

Then

$$\{B_{\varepsilon_a}(a) : a \in A\}$$

is an open cover of $A$ (if $A$ is uncountable, it will be an uncountable open cover). Since $A$ is compact, there is a finite subcover

$$\left\{B_{\varepsilon_{a_1}}(a_1), \ldots, B_{\varepsilon_{a_m}}(a_m)\right\}$$

Then

$$\begin{aligned} \mathbf{N} &= \{n : x_n \in A\} \\ &\subseteq \left\{n : x_n \in \left(B_{\varepsilon_{a_1}}(a_1) \cup \cdots \cup B_{\varepsilon_{a_m}}(a_m)\right)\right\} \\ &= \{n : x_n \in B_{\varepsilon_{a_1}}(a_1)\} \cup \cdots \cup \{n : x_n \in B_{\varepsilon_{a_m}}(a_m)\} \end{aligned}$$

so $\mathbf{N}$ is contained in a finite union of sets, each of which is finite by Equation (1). Thus, $\mathbf{N}$ must be finite, a contradiction which proves that $A$ is sequentially compact.

For the converse, see de la Fuente. $\square$

# Totally Bounded Sets

**Definition 4.** *A set $A$ in a metric space $(X, d)$ is* totally bounded *if, for every $\varepsilon > 0$,*

$$\exists x_1, \ldots, x_n \in A \text{ s.t. } A \subseteq \cup_{i=1}^{n} B_\varepsilon(x_i)$$

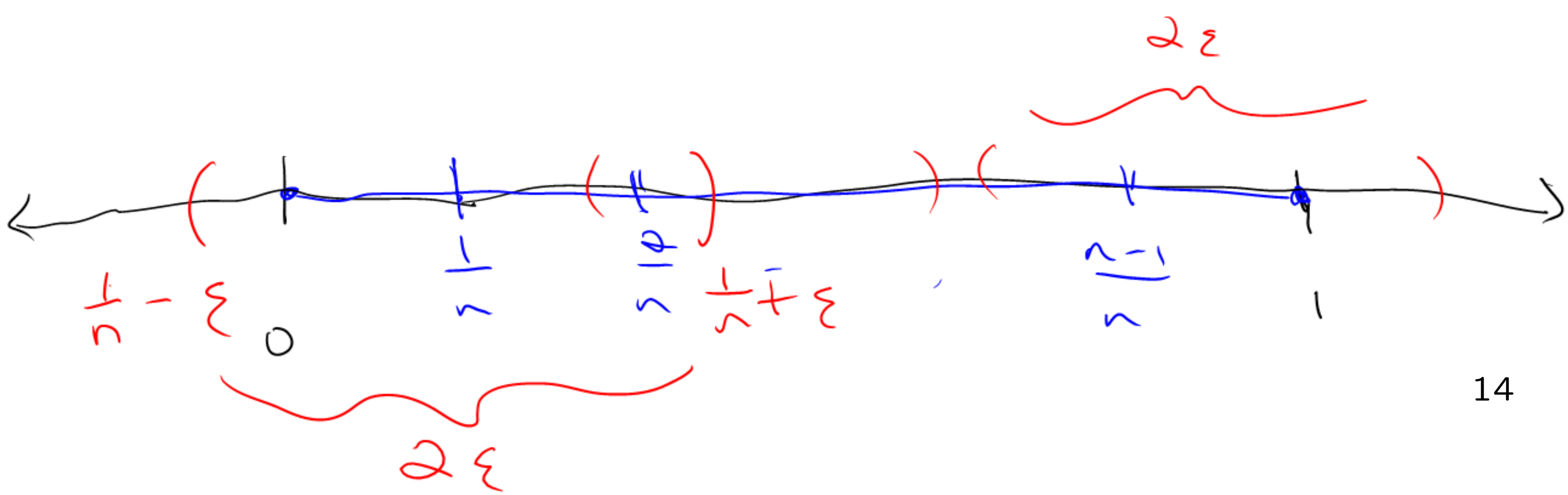
# Totally Bounded Sets

**Example:** Take $A = [0,1]$ with the Euclidean metric. Given $\varepsilon > 0$, let $n > \frac{1}{\varepsilon}$. Then we may take

$\varepsilon > \frac{1}{n}$

$$x_1 = \frac{1}{n}, x_2 = \frac{2}{n}, \ldots, x_{n-1} = \frac{n-1}{n}$$

Then $[0,1] \subset \cup_{k=1}^{n-1} B_\varepsilon(\frac{k}{n})$.



# Totally Bounded Sets

**Example:** Consider $X = [0, 1]$ with the discrete metric

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

$X$ is not totally bounded. To see this, take $\varepsilon = \frac{1}{2}$. Then for any $x$, $B_\varepsilon(x) = \{x\}$, so given any finite set $x_1, \ldots, x_n$,

$$\cup_{i=1}^{n} B_\varepsilon(x_i) = \{x_1, \ldots, x_n\} \not\supseteq [0, 1]$$

However, $X$ is bounded because $X = B_2(0)$.

# Totally Bounded Sets

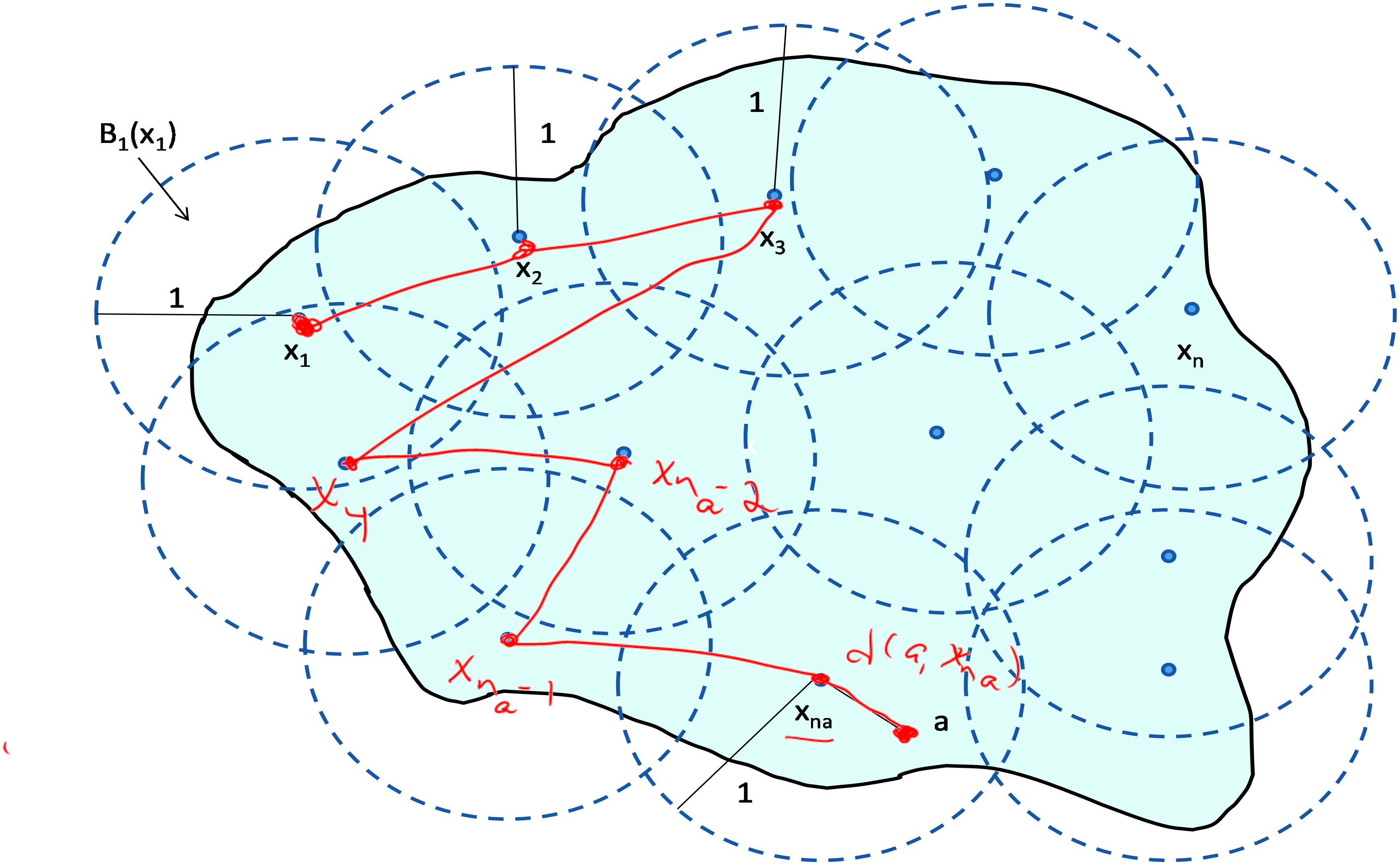
Note that any totally bounded set in a metric space $(X, d)$ is also bounded. To see this, let $A \subset X$ be totally bounded. Then $\exists x_1, \ldots, x_n \in A$ such that $A \subset B_1(x_1) \cup \cdots \cup B_1(x_n)$. Let

$$M = 1 + d(x_1, x_2) + \cdots + d(x_{n-1}, x_n)$$

Then $M < \infty$. Now fix $a \in A$. We claim $d(a, x_1) < M$. To see this, notice that there is some $n_a \in \{1, \ldots, n\}$ for which $a \in B_1(x_{n_a})$. Then

$$\begin{aligned} d(a, x_1) &\leq d(a, x_{n_a}) + \sum_{k=1}^{n} d(x_k, x_{k+1}) \\ &< 1 + \sum_{k=1}^{n} d(x_k, x_{k+1}) \\ &= M \end{aligned}$$

$$d(a, x_1) = ? \leq d(a, x_{n_a}) + \sum_{k=1}^{n_a} d(x_k, x_{k+1})$$




$a \in B_1(x_{n_a})$

# Totally Bounded Sets

**Remark 4.** Every compact subset of a metric space is totally bounded:

Fix $\varepsilon$ and consider the open cover

$$\mathcal{U}_\varepsilon = \{B_\varepsilon(a) : a \in A\}$$

If $A$ is compact, then every open cover of $A$ has a finite subcover; in particular, $\mathcal{U}_\varepsilon$ must have a finite subcover, but this just says that $A$ is totally bounded.

$\Rightarrow \exists\, a_1, \ldots, a_n$ s.t.

$$A \subseteq B_\varepsilon(a_1) \cup \cdots \cup B_\varepsilon(a_n)$$

# Compactness and Totally Bounded Sets

**Theorem 5** (Thm. 8.16)**.** *Let $A$ be a subset of a metric space $(X, d)$. Then $A$ is compact if and only if it is complete and totally bounded.*

*Proof.* Here is a sketch of the proof; see de la Fuente for details. Compact implies totally bounded (Remark 4). Suppose $\{x_n\}$ is a Cauchy sequence in $A$. Since $A$ is compact, $A$ is sequentially compact, hence $\{x_n\}$ has a convergent subsequence $x_{n_k} \to a \in A$. Since $\{x_n\}$ is Cauchy, $x_n \to a$ (by Problem 3 of Problem Set 2), so $A$ is complete.

Conversely, suppose $A$ is complete and totally bounded. Let $\{x_n\}$ be a sequence in $A$. Because $A$ is totally bounded, we can extract a Cauchy subsequence $\{x_{n_k}\}$ (why?). Because $A$ is complete, $x_{n_k} \to a$ for some $a \in A$, which shows that $A$ is sequentially compact and hence compact. $\square$

# Compact $\iff$ Closed and Totally Bounded

Putting these together:

**Corollary 1.** *Let $A$ be a subset of a complete metric space $(X, d)$. Then $A$ is compact if and only if $A$ is closed and totally bounded.*

$A \subseteq X$ , $(X, d)$ is complete

$$\begin{aligned} A \text{ compact} &\Rightarrow A \text{ complete and totally bounded} \\ &\Rightarrow A \text{ closed and totally bounded} \\ A \text{ closed and totally bounded} &\Rightarrow A \text{ complete and totally bounded} \\ &\Rightarrow A \text{ compact} \end{aligned}$$

**Example:** $[0,1]$ is compact in $\mathbf{E}^1$.

**Note:** compact $\Rightarrow$ closed and bounded, but converse need not be true.

E.g. $[0,1]$ with the discrete metric.

# Heine-Borel Theorem - $\mathbf{E}^1$

**Theorem 6** (Thm. 8.19, Heine-Borel)**.** *If* $A \subseteq \mathbf{E}^1$*, then* $A$ *is compact if and only if* $A$ *is closed and bounded.*

*Proof.* Let $A$ be a closed, bounded subset of $\mathbf{R}$. Then $A \subseteq [a, b]$ for some interval $[a, b]$. Let $\{x_n\}$ be a sequence of elements of $[a, b]$. By the Bolzano-Weierstrass Theorem, $\{x_n\}$ contains a convergent subsequence with limit $x \in \mathbf{R}$. Since $[a, b]$ is closed, $x \in [a, b]$. Thus, we have shown that $[a, b]$ is sequentially compact, hence compact. $A$ is a closed subset of $[a, b]$, hence $A$ is compact.

Conversely, if $A$ is compact, $A$ is closed and bounded. $\square$

# Heine-Borel Theorem - $\mathbf{E}^n$

**Theorem 7** (Thm. 8.20, Heine-Borel)**.** *If $A \subseteq \mathbf{E}^n$, then $A$ is compact if and only if $A$ is closed and bounded.*

*Proof.* See de la Fuente. □

**Example:** The closed interval

$$[a, b] = \{x \in \mathbf{R}^n : a_i \leq x_i \leq b_i \text{ for each } i = 1, \ldots, n\}$$

is compact in $\mathbf{E}^n$ for any $a, b \in \mathbf{R}^n$.

# Continuous Images of Compact Sets

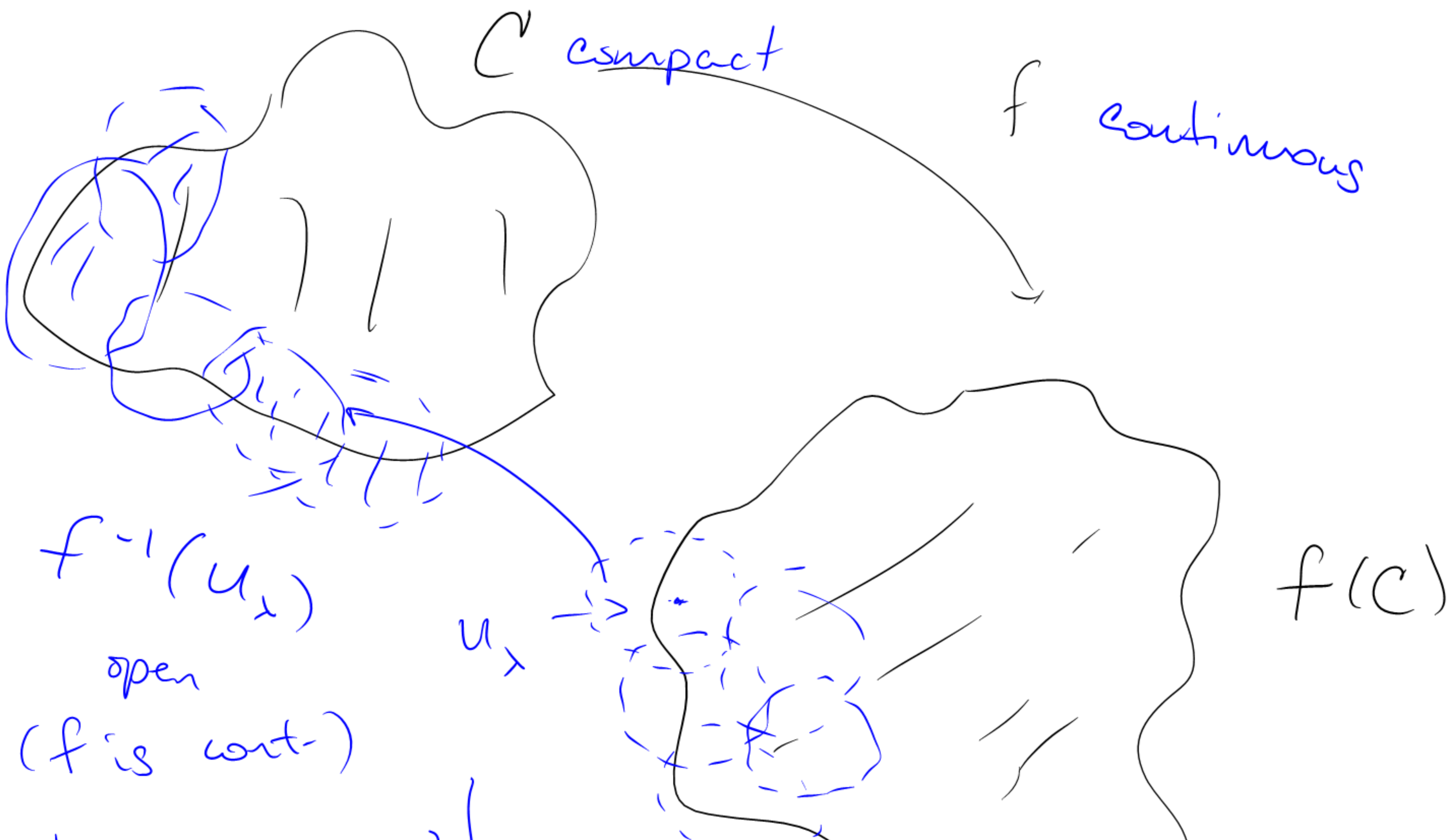
**Theorem 8** (8.21). *Let $(X,d)$ and $(Y,\rho)$ be metric spaces. If $f : X \to Y$ is continuous and $C$ is a compact subset of $(X,d)$, then $f(C)$ is compact in $(Y,\rho)$.*

*Proof.* There is a proof in de la Fuente using sequential compactness. Here we give an alternative proof using directly the open cover definition of compactness.

Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover of $f(C)$. For each point $c \in C$, $f(c) \in f(C)$ so $f(c) \in U_{\lambda_c}$ for some $\lambda_c \in \Lambda$, that is, $c \in f^{-1}\left(U_{\lambda_c}\right)$. Thus the collection $\left\{f^{-1}(U_\lambda) : \lambda \in \Lambda\right\}$ is a cover of $C$; in addition, since $f$ is continuous, each set $f^{-1}(U_\lambda)$ is open in $C$, so $\left\{f^{-1}\left(U_\lambda\right):\lambda\in\Lambda\right\}$ is an open cover of $C$. Since $C$ is compact, there is a finite subcover

$$\left\{f^{-1}\left(U_{\lambda_1}\right),\ldots,f^{-1}\left(U_{\lambda_n}\right)\right\}$$

of $C$. Given $x\in f(C)$, there exists $c\in C$ such that $f(c)=x$, and $c\in f^{-1}\left(U_{\lambda_i}\right)$ for some $i$, so $x\in U_{\lambda_i}$. Thus, $\{U_{\lambda_1},\ldots,U_{\lambda_n}\}$ is a finite subcover of $f(C)$, so $f(C)$ is compact. $\square$




$\{f^{-1}(U_\lambda) : \lambda \in \Lambda\}$ covers C

$\{U_\lambda : \lambda \in \Lambda\}$ open cover of f(C)

# Extreme Value Theorem

**Corollary 2** (Thm. 8.22, Extreme Value Theorem)**.** *Let $C$ be a compact set in a metric space $(X, d)$, and suppose $f : C \to \mathbf{R}$ is continuous. Then $f$ is bounded on $C$ and attains its minimum and maximum on $C$.*

*Proof.* $f(C)$ is compact by Theorem 8.21, hence closed and bounded. Let $M = \sup f(C)$; $M < \infty$. Then there exists $y_m \in f(C)$ such that

$$M - \frac{1}{m} \leq y_m \leq M$$

so $M$ is a limit point of $f(C)$. Since $f(C)$ is closed, $M \in f(C)$, i.e. there exists $c \in C$ such that $f(c) = M = \sup f(C)$, so $f$ attains its maximum at $c$. The proof for the minimum is similar. □

# Compactness and Uniform Continuity

**Theorem 9** (Thm. 8.24)**.** *Let* $(X, d)$ *and* $(Y, \rho)$ *be metric spaces,* $C$ *a compact subset of* $X$*, and* $f : C \to Y$ *continuous. Then* $f$ *is uniformly continuous on* $C$*.*

*Proof.* Fix $\varepsilon > 0$. We ignore $X$ and consider $f$ as defined on the metric space $(C, d)$. Given $c \in C$, find $\delta(c) > 0$ such that

$$x \in C,\ d(x, c) < 2\delta(c) \Rightarrow \rho(f(x), f(c)) < \frac{\varepsilon}{2}$$

f is continuous

Let

$$U_c = B_{\delta(c)}(c)$$

open

Then

$$\{U_c : c \in C\}$$

is an open cover of $C$. Since $C$ is compact, there is a finite subcover

$$\{U_{c_1}, \ldots, U_{c_n}\}$$

Let

$$\delta = \min\{\delta(c_1), \ldots, \delta(c_n)\} > 0$$

Given $x, y \in C$ with $d(x, y) < \delta$, note that $x \in U_{c_i}$ for some $i \in \{1, \ldots, n\}$, so $d(x, c_i) < \delta(c_i)$.

$$\begin{aligned} d(y, c_i) &\leq d(y, x) + d(x, c_i) \\ &< \delta + \delta(c_i) \\ &\leq \delta(c_i) + \delta(c_i) \\ &= 2\delta(c_i) \end{aligned}$$

$$d(x, c_i) < \delta(c_i) \leq 2\delta(c_i)$$

so

$$\begin{aligned}\rho(f(x),f(y)) &\le \rho(f(x),f(c_i))+\rho(f(c_i),f(y))\\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}\\ &= \varepsilon\end{aligned}$$

which proves that $f$ is uniformly continuous. $\square$