Econ 204 2010

Lecture 9

Outline

1. Quotient Vector Spaces
2. Matrix Representations of Linear Transformations
3. Change of Basis and Similarity
4. Eigenvalues and Eigenvectors
5. Diagonalization
Quotient Vector Spaces

Given a vector space $X$ and a vector subspace $W$ of $X$, define an equivalence relation by

$$x \sim y \iff x - y \in W$$

Form a new vector space $X/W$: the set of vectors is

$$\{[x] : x \in X\}$$

where $[x]$ denotes the equivalence class of $x$ with respect to $\sim$.

$X/W$ is read “$X$ mod $W$”.

Note that the vectors in $X/W$ are sets of vectors in $X$: for $x \in X$,

$$[x] = \{x + w : w \in W\}$$
Quotient Vector Spaces

We claim that $X/W$ can be viewed as a vector space over $F$. Define the vector space operations $+$, $\cdot$ in $X/W$ as follows:

Define

$$[x] + [y] = [x + y]$$

$$\alpha[x] = [\alpha x]$$

Exercise: Verify that $\sim$ is an equivalence relation and that vector addition and scalar multiplication are well-defined.

Then $X/W$ is a vector space over $F$ with these definitions for $+$ and $\cdot$. 
Quotient Vector Spaces

Example: Let $X = \mathbb{R}^3$ and let $W = \{x \in \mathbb{R}^3 : x_1 = x_2 = 0\}$. Then for $x, y \in \mathbb{R}^3$,

$$x \sim y \iff x - y \in W$$

$$\iff x_1 - y_1 = 0, x_2 - y_2 = 0$$

$$\iff x_1 = y_1, x_2 = y_2$$

and

$$[x] = \{x + w : w \in W\} = \{(x_1, x_2, z) : z \in \mathbb{R}\}$$

So the equivalence class corresponding to $x$ is the line in $\mathbb{R}^3$ through $x$ parallel to the axis of the third coordinate.
Example, cont.

What is $X/W$? Intuitively this equivalence relation ignores the third coordinate, and we can identify the equivalence class $[x]$ with the vector $(x_1, x_2) \in \mathbb{R}^2$.

The next two results show how to formalize this connection.
Theorem 1. If $X$ is a vector space with $\dim X = n$ for some $n \in \mathbb{N}$ and $W$ is a vector subspace of $X$, then

$$\dim(X/W) = \dim X - \dim W$$

Proof. (Sketch) Begin with a basis $\{w_1, \ldots, w_c\}$ for $W$, and a basis $\{[x_1], \ldots, [x_k]\}$ for $X/W$. Show that

$$\{w_1, \ldots, w_c\} \cup \{x_1, \ldots, x_k\}$$

is a basis for $X$. \qed
Quotient Vector Spaces

Theorem 2. Let $X$ and $Y$ be vector spaces over the same field $F$ and $T \in L(X, Y)$. Then $\text{Im} T$ is isomorphic to $X/\ker T$.

Proof. Notice that if $X$ is finite-dimensional, then

$$\dim(X/\ker T) = \dim X - \dim \ker T \quad \text{(by the previous theorem)}$$

$$= \text{Rank } T \quad \text{(by the Rank-Nullity Theorem)}$$

$$= \dim \text{Im } T$$

so $X/\ker T$ is isomorphic to $\text{Im } T$. (why??)

We prove that this is true in general, and that the isomorphism is natural.
Define

$$\tilde{T}([x]) = T(x)$$

We first need to check that this is well-defined, that is, that if $[x] = [x']$ then $\tilde{T}([x]) = \tilde{T}([x'])$.

$$[x] = [x'] \implies x \sim x'$$
$$\implies x - x' \in \ker T$$
$$\implies T(x - x') = 0$$
$$\implies T(x) = T(x')$$

so $\tilde{T}$ is well-defined.

Clearly, $\tilde{T} : X/\ker T \to \Im T$. It is easy to check that $\tilde{T}$ is linear,
so $\tilde{T} \in L(X/\ker T, \text{Im } T)$. Next we show that $\tilde{T}$ is an isomorphism.

$$\tilde{T}([x]) = \tilde{T}([y]) \implies T(x) = T(y)$$

$$\implies T(x - y) = 0$$

$$\implies x - y \in \ker T$$

$$\implies x \sim y$$

$$\implies [x] = [y]$$

so $\tilde{T}$ is one-to-one.

$$y \in \text{Im } T \implies \exists x \in X \text{ s.t. } T(x) = y$$

$$\implies \tilde{T}([x]) = y$$

so $\tilde{T}$ is onto, hence $\tilde{T}$ is an isomorphism.
**Example:** Consider $T \in L(\mathbb{R}^3, \mathbb{R}^2)$ defined by

$$T(x_1, x_2, x_3) = (x_1, x_2)$$

Then

$$\ker T = \{x \in \mathbb{R}^3 : x_1 = x_2 = 0\}$$

is the $x_3$-axis.

Given $x$, the equivalence class $[x]$ is just the line through $x$ parallel to the $x_3$-axis.

$$\tilde{T}([x]) = T(x_1, x_2, x_3) = (x_1, x_2)$$

and

$$\text{Im } T = \mathbb{R}^2, \quad X/\ker T \cong \mathbb{R}^2 = \text{Im } T$$

as we suggested intuitively above (here the symbol $\cong$ denotes isomorphism, that is, we write $Y \cong Z$ if $Y$ and $Z$ are isomorphic.)
Coordinate Representations

Every real vector space $X$ with dimension $n$ is isomorphic to $\mathbb{R}^n$. What’s the isomorphism?

Let $X$ be a finite-dimensional vector space over $\mathbb{R}$ with $\dim X = n$. Fix any Hamel basis $V = \{v_1, \ldots, v_n\}$ of $X$. Any $x \in X$ has a unique representation

$$x = \sum_{j=1}^{n} \beta_j v_j$$

(here, we allow $\beta_j = 0$).

$$crd_V(x) = \left( \begin{array}{c} \beta_1 \\ \vdots \\ \beta_n \end{array} \right) \in \mathbb{R}^n$$
$\text{crd}_V(x)$ is the vector of coordinates of $x$ with respect to the basis $V$.

\[
\text{crd}_V(v_1) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad \text{crd}_V(v_2) = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad \text{crd}_V(v_n) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}
\]

$\text{crd}_V$ is an isomorphism from $X$ to $\mathbb{R}^n$. 
Matrix Representations of Linear Transformations

Suppose $T \in L(X, Y)$, dim $X = n$, dim $Y = m$. Fix bases

$V = \{v_1, \ldots, v_n\}$ of $X$

$W = \{w_1, \ldots, w_m\}$ of $Y$

$T(v_j) \in Y$, so

$$T(v_j) = \sum_{i=1}^{m} \alpha_{ij}w_i$$

Define

$$Mtx_{W,V}(T) = \begin{pmatrix}
\alpha_{11} & \cdots & \alpha_{1n} \\
\vdots & \ddots & \vdots \\
\alpha_{m1} & \cdots & \alpha_{mn}
\end{pmatrix}$$
Matrix Representations of Linear Transformations

Notice that the columns are the coordinates (expressed with respect to $W$) of $T(v_1), \ldots, T(v_n)$.

Observe

$$
\begin{pmatrix}
\alpha_{11} & \cdots & \alpha_{1n} \\
\vdots & \ddots & \vdots \\
\alpha_{m1} & \cdots & \alpha_{mn}
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
= 
\begin{pmatrix}
\alpha_{11} \\
\vdots \\
\alpha_{m1}
\end{pmatrix}
$$

so

$$
Mtx_{W,V}(T) \cdot \text{crd}_V(v_j) = \text{crd}_W(T(v_j)) \\
Mtx_{W,V}(T) \cdot \text{crd}_V(x) = \text{crd}_W(T(x)) \quad \forall x \in X
$$
Matrix Representations

Multiplying a vector by a matrix does two things:

- Computes the action of $T$

- Accounts for the change in basis
Example: $X = Y = \mathbb{R}^2$, $V = \{(1, 0), (0, 1)\}$, $W = \{(1, 1), (-1, 1)\}$, $T = id$, that is, $T(x) = x$ for each $x$.

$$Mtx_{W,V}(T) \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$Mtx_{W,V}(T)$ is the matrix that changes basis from $V$ to $W$. 
How do we compute it?

\[ v_1 = (1, 0) = \alpha_{11}(1, 1) + \alpha_{21}(-1, 1) \]
\[ \alpha_{11} - \alpha_{21} = 1 \]
\[ \alpha_{11} + \alpha_{21} = 0 \]
\[ 2\alpha_{11} = 1, \quad \alpha_{11} = \frac{1}{2} \]
\[ \alpha_{21} = -\frac{1}{2} \]

\[ v_2 = (0, 1) = \alpha_{12}(1, 1) + \alpha_{22}(-1, 1) \]
\[ \alpha_{12} - \alpha_{22} = 0 \]
\[ \alpha_{12} + \alpha_{22} = 1 \]
\[ 2\alpha_{12} = 1, \quad \alpha_{12} = \frac{1}{2} \]
\[ \alpha_{22} = \frac{1}{2} \]
So

\[ Mtx_{W,V}(id) = \left( \begin{array}{cc} 1/2 & 1/2 \\ -1/2 & 1/2 \end{array} \right) \]
Matrix Representations

**Theorem 3** (Thm. 3.5’). Let $X$ and $Y$ be vector spaces over the same field $F$, with $\dim X = n$, $\dim Y = m$. Then $L(X,Y)$, the space of linear transformations from $X$ to $Y$, is isomorphic to $F_{m \times n}$, the vector space of $m \times n$ matrices over $F$. If $V = \{v_1, \ldots, v_n\}$ is a basis for $X$ and $W = \{w_1, \ldots, w_m\}$ is a basis for $Y$, then

$$Mtx_{W,V} \in L(L(X,Y), F_{m \times n})$$

and $Mtx_{W,V}$ is an isomorphism from $L(X,Y)$ to $F_{m \times n}$. 
Matrix Representations

**Theorem 4** (From Handout). *Let* $X, Y, Z$ *be finite-dimensional vector spaces with bases* $U, V, W$ *respectively. Let* $S \in L(X, Y)$ *and* $T \in L(Y, Z)$. *Then*

$$Mtx_{W,V}(T) \cdot Mtx_{V,U}(S) = Mtx_{W,U}(T \circ S)$$

*i.e. matrix multiplication corresponds via the matrix representation isomorphism to composition of linear transformations.*

**Proof.** See handout. □

Note that $Mtx_{W,V}$ is a function from $L(X, Y)$ to the space $F_{m \times n}$ of $m \times n$ matrices, while $Mtx_{W,V}(T)$ is an $m \times n$ matrix.
Matrix Representations

The theorem can be summarized by the following “Commutative Diagram:"

\[
\begin{array}{ccc}
  X & \rightarrow & Y \\
  \downarrow \text{crd}_U & & \downarrow \text{crd}_V \\
  \mathbb{R}^n & \rightarrow & \mathbb{R}^m \\
  \text{Mtx}_{V,U}(S) & & \text{Mtx}_{W,V}(T) \\
  \end{array}
\]

We say the diagram commutes because you get the same answer any way you go around the diagram (in directions allowed by the arrows). The \( \text{crd} \) arrows go in both directions because \( \text{crd} \) is an isomorphism.
Change of Basis

Let $X$ be a finite-dimensional vector space with basis $V$. If $T \in L(X, X)$ it is customary to use the same basis in the domain and range. In this case, $Mtx_V(T)$ denotes $Mtx_{V, V}(T)$.

**Question:** If $W$ is another basis for $X$, how are $Mtx_V(T)$ and $Mtx_W(T)$ related?
\[ M_{tx\, V,W}(id) \cdot M_{tx\, W}(T) \cdot M_{tx\, W,V}(id) = M_{tx\, V,W}(id) \cdot M_{tx\, W,V}(T \circ id) \]
\[ = M_{tx\, V,V}(id \circ T \circ id) \]
\[ = M_{tx\, V}(T) \]

and

\[ M_{tx\, V,W}(id) \cdot M_{tx\, W,V}(id) = M_{tx\, V,V}(id) \]
\[ = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix} \]
So this says that

\[ Mtx_V(T) = P^{-1}Mtx_W(T)P \]

for the invertible matrix

\[ P = Mtx_{W,V}(id) \]

that is the change of basis matrix.

On the other hand, if \( P \) is any invertible matrix, then \( P \) is also a change of basis matrix for appropriate corresponding bases (see handout).


**Similarity**

**Definition 1.** *Square matrices* $A$ and $B$ are similar *if*

$$A = P^{-1}BP$$

*for some invertible matrix* $P$. 
Similarity

Theorem 5. Suppose that $X$ is a finite-dimensional vector space.

1. If $T \in L(X, X)$ then any two matrix representations of $T$ are similar. That is, if $U, W$ are any two bases of $X$, then $Mtx_W(T)$ and $Mtx_U(T)$ are similar.

2. Conversely, two similar matrices represent the same linear transformation $T$, relative to suitable bases. That is, given similar matrices $A, B$ with $A = P^{-1}BP$ and any basis $U$, there is a basis $W$ and $T \in L(X, X)$ such that

$$
B = Mtx_U(T) \\
A = Mtx_W(T) \\
P = Mtx_{U,W}(id) \\
P^{-1} = Mtx_{W,U}(id)
$$
Proof. See Handout on Diagonalization and Quadratic Forms.
Eigenvalues and Eigenvectors

Here, we define eigenvalues and eigenvectors of a linear transformation and show that $\lambda$ is an eigenvalue of $T$ if and only if $\lambda$ is an eigenvalue for some matrix representation of $T$ if and only if $\lambda$ is an eigenvalue for every matrix representation of $T$.

**Definition 2.** Let $X$ be a vector space and $T \in L(X, X)$. We say that $\lambda$ is an eigenvalue of $T$ and $\nu \neq 0$ is an eigenvector corresponding to $\lambda$ if $T(\nu) = \lambda \nu$. 
Eigenvalues and Eigenvectors

Theorem 6 (Theorem 4 in Handout). Let $X$ be a finite-dimensional vector space, and $U$ a basis. Then $\lambda$ is an eigenvalue of $T$ if and only if $\lambda$ is an eigenvalue of $Mtx_U(T)$. $v$ is an eigenvector of $T$ corresponding to $\lambda$ if and only if $crd_U(v)$ is an eigenvector of $Mtx_U(T)$ corresponding to $\lambda$.

Proof. By the Commutative Diagram Theorem,

\[
T(v) = \lambda v \iff crd_U(T(v)) = crd_U(\lambda v)
\]

\[
\iff Mtx_U(T)(crd_U(v)) = \lambda(crd_U(v))
\]

\[\square\]
Computing Eigenvalues and Eigenvectors

Suppose \( \text{dim } X = n \); let \( I \) be the \( n \times n \) identity matrix. Given \( T \in L(X, X) \), fix a basis \( U \) and let

\[
A = Mtx_U(T)
\]

Find the eigenvalues of \( T \) by computing the eigenvalues of \( A \):

\[
Av = \lambda v \iff (A - \lambda I)v = 0
\]

\[
\iff (A - \lambda I) \text{ is not invertible}
\]

\[
\iff \det(A - \lambda I) = 0
\]
We have the following facts:

- If $A \in \mathbf{R}_{n \times n}$,
  
  $$f(\lambda) = \det(A - \lambda I)$$

  is an $n^{th}$ degree polynomial in $\lambda$ with real coefficients; it is called the characteristic polynomial of $A$.

- $f$ has $n$ roots in $\mathbf{C}$, counting multiplicity:

  $$f(\lambda) = (\lambda - c_1)(\lambda - c_2) \cdots (\lambda - c_n)$$

  where $c_1, \ldots, c_n \in \mathbf{C}$ are the eigenvalues; the $c_j$’s are not necessarily distinct. Notice that $f(\lambda) = 0$ if and only if $\lambda \in \{c_1, \ldots, c_n\}$, so the roots are the solutions of the equation $f(\lambda) = 0$. 

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• the roots that are not real come in conjugate pairs:

\[ f(a + bi) = 0 \Leftrightarrow f(a - bi) = 0 \]

• if \( \lambda = c_j \in \mathbb{R} \), there is a corresponding eigenvector in \( \mathbb{R}^n \).

• if \( \lambda = c_j \notin \mathbb{R} \), the corresponding eigenvectors are in \( \mathbb{C}^n \setminus \mathbb{R}^n \).
Diagonalization

**Definition 3.** Suppose $X$ is a finite-dimensional vector space with basis $U$. Given a linear transformation $T \in L(X, X)$, let

$$A = Mtx_U(T)$$

We say that $A$ can be diagonalized if there is a basis $W$ for $X$ such that $Mtx_W(T)$ is a diagonal matrix, that is,
\[
Mtx_W(T) = \begin{pmatrix}
\lambda_1 & 0 & 0 & \cdots & 0 & 0 \\
0 & \lambda_2 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \lambda_n \\
\end{pmatrix}
\]

Notice that the eigenvectors of \( Mtx_W(T) \) are exactly the standard basis vectors of \( \mathbb{R}^n \). But \( w_j \) is an eigenvector of \( T \) corresponding to \( \lambda_j \) if and only if \( crd_W(w_j) \) is an eigenvector of \( Mtx_W(T) \), and \( crd_W(w_j) \) is the \( j^{th} \) standard basis vector of \( \mathbb{R}^n \), so \( W = \{w_1, \ldots, w_n\} \) where \( w_j \) is an eigenvector corresponding to \( \lambda_j \).

Then the action of \( T \) is clear: it stretches each basis element \( w_i \) by the factor \( \lambda_i \).
Diagonalization

**Theorem 7 (Thm. 6.7’).** Let $X$ be an $n$-dimensional vector space, $T \in L(X, X)$, $U$ any basis of $X$, and $A = Mtx_U(T)$. Then the following are equivalent:

1. $A$ can be diagonalized

2. there is a basis $W$ for $X$ consisting of eigenvectors of $T$

3. there is a basis $V$ for $\mathbb{R}^n$ consisting of eigenvectors of $A$

*Proof.* Follows from Theorem 6.7 in de la Fuente and Theorem 4 from the Handout. \[\square\]
Diagonalization

Theorem 8 (Thm. 6.8'). Let $X$ be a vector space and $T \in L(X, X)$.

1. If $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of $T$ with corresponding eigenvectors $v_1, \ldots, v_m$, then $\{v_1, \ldots, v_m\}$ is linearly independent.

2. If $\dim X = n$ and $T$ has $n$ distinct eigenvalues, then $X$ has a basis consisting of eigenvectors of $T$; consequently, if $U$ is any basis of $X$, then $Mtx_U(T)$ is diagonalizable.

Proof. This is an adaptation of the proof of Theorem 6.8 in de la Fuente.