Economics 204 Summer/Fall 2010
Lecture 7–Tuesday August 3, 2010

Section 2.9. Connected Sets

Definition 1 Two sets A, B in a metric space are *separated* if

$$\bar{A} \cap B = A \cap \bar{B} = \emptyset$$

A set in a metric space is *connected* if it cannot be written as the union of two nonempty separated sets.

Remark: In other texts, you will see the following equivalent definition: A set Y in a metric space X is connected if there do not exist open sets A and B such that $A \cap B = \emptyset$, $Y \subseteq A \cup B$ and $A \cap Y \neq \emptyset$ and $B \cap Y \neq \emptyset$.

Example: $[0, 1)$ and $[1, 2]$ are disjoint but not separated:

$$\overline{[0,1)} \cap [1,2] = [0,1] \cap [1,2] = \{1\} \neq \emptyset$$

$[0, 1)$ and $(1, 2]$ are separated:

$$\begin{aligned} \overline{[0,1)} \cap (1,2] &= [0,1] \cap (1,2] = \emptyset \\ [0,1) \cap \overline{(1,2]} &= [0,1) \cap [1,2] = \emptyset \end{aligned}$$

Note that $d([0, 1), (1, 2]) = 0$ even though the sets are separated. Note also that separation does *not* require that $\bar{A} \cap \bar{B} = \emptyset$. For example,

$$[0, 1) \cup (1, 2]$$

is not connected, since $[0, 1)$ and $(1, 2]$ are separated.

In $\mathbf{R}$, connected sets are easy to characterize: they are simply intervals.

Theorem 2 (Thm. 9.2) *A set $S \subseteq \mathbf{E}^1$ of real numbers is connected if and only if it is an interval, i.e. if $x, y \in S$ and $z \in (x, y)$, then $z \in S$.*

Proof: First, we show that if S is connected then S is an interval. We do this by proving the contrapositive: if S is not an interval, then it is not connected. If S is not an interval, find

$$x, y \in S \text{ and } z \notin S \text{ s.t. } x < z < y$$

Let

$$A = S \cap (-\infty, z), \ B = S \cap (z, \infty)$$

Then

$$
\begin{aligned}
\bar{A} \cap B &\subseteq \overline{(-\infty, z)} \cap (z, \infty) = (-\infty, z] \cap (z, \infty) = \emptyset \\
A \cap \bar{B} &\subseteq (-\infty, z) \cap \overline{(z, \infty)} = (-\infty, z) \cap [z, \infty) = \emptyset \\
A \cup B &= (S \cap (-\infty, z)) \cup (S \cap (z, \infty)) \\
&= S \setminus \{z\} \\
&= S \\
x &\in A, \text{ so } A \neq \emptyset \\
y &\in B, \text{ so } B \neq \emptyset
\end{aligned}
$$

So S is not connected. We have shown that if S is not an interval, then S is not connected; therefore, if S is connected, then S is an interval.

Now, we need to show that if S is an interval, then it is connected. This is much like the proof of the Intermediate Value Theorem. See de la Fuente for the details.■

Notice that this result is only valid in $\mathbf{R}$. For example, connected sets in $\mathbf{R}^n$ need not be intervals, or even convex.

In a general metric space, continuity will preserve connectedness. You can view this result as a generalization of the Intermediate Value Theorem, to which we return below.

Theorem 3 (Thm. 9.3) *Let X be a metric space and $f : X \to Y$ be continuous. If C is a connected subset of X, then $f(C)$ is connected.*

Proof: We prove the contrapositive: if $f(C)$ is not connected, then C is not connected. Suppose $f(C)$ is not connected. Then there exist P, Q such that $P \neq \emptyset \neq Q$, $f(C) = P \cup Q$, and

$$\bar{P} \cap Q = P \cap \bar{Q} = \emptyset$$

Let

$$A = f^{-1}(P) \cap C \text{ and } B = f^{-1}(Q) \cap C$$

(See Figure 1.)

Then

$$
\begin{aligned}
A \cup B &= \left(f^{-1}(P) \cap C\right) \cup \left(f^{-1}(Q) \cap C\right) \\
&= \left(f^{-1}(P) \cup f^{-1}(Q)\right) \cap C \\
&= f^{-1}(P \cup Q) \cap C \\
&= f^{-1}(f(C)) \cap C \\
&= C
\end{aligned}
$$

Also, $A = f^{-1}(P) \cap C \neq \emptyset$ and $B = f^{-1}(Q) \cap C \neq \emptyset$. Then note

$$A = f^{-1}(P) \cap C \subseteq f^{-1}(P) \subseteq f^{-1}\left(\bar{P}\right)$$

Since f is continuous, $f^{-1}(\bar{P})$ is closed, so

$$\bar{A} \subseteq f^{-1}\left(\bar{P}\right)$$

Similarly,

$$B = f^{-1}(Q) \cap C \subseteq f^{-1}(Q) \subseteq f^{-1}\left(\bar{Q}\right)$$

and $f^{-1}(\bar{Q})$ is closed, so

$$\bar{B} \subseteq f^{-1}\left(\bar{Q}\right)$$

Then

$$\begin{aligned} \bar{A} \cap B &\subseteq f^{-1}\left(\bar{P}\right) \cap f^{-1}(Q) \\ &= f^{-1}\left(\bar{P} \cap Q\right) \\ &= f^{-1}(\emptyset) \\ &= \emptyset \end{aligned}$$

and similarly

$$\begin{aligned} A \cap \bar{B} &\subseteq f^{-1}(P) \cap f^{-1}\left(\bar{Q}\right) \\ &= f^{-1}\left(P \cap \bar{Q}\right) \\ &= f^{-1}(\emptyset) \\ &= \emptyset \end{aligned}$$

So C is not connected. We have shown that $f(C)$ not connected implies C not connected; therefore, C connected implies $f(C)$ connected. ■

As noted above, this is essentially the same principle as the Intermediate Value Theorem; the stronger properties of $\mathbf{R}$ yield a stronger result. This observation lets us give a third, and slickest, proof of the Intermediate Value Theorem. It is short because a substantial part of the argument was incorporated into the proof that $C \subseteq \mathbf{R}$ is connected if and only if C is an interval, and the proof that if C is connected, then $f(C)$ is connected. Here's the proof:

Corollary 4 (Intermediate Value Theorem) *If $f : [a, b] \to \mathbf{R}$ is continuous, and $f(a) < d < f(b)$, then there exists $c \in (a, b)$ such that $f(c) = d$.*

Proof: Since $[a, b]$ is an interval, it is connected. So $f([a, b])$ is connected, hence $f([a, b])$ is an interval. $f(a) \in f([a, b])$, and $f(b) \in f([a, b])$, and $d \in [f(a), f(b)]$; since $f([a, b])$ is an interval, $d \in f([a, b])$, i.e. there exists $c \in [a, b]$ such that $f(c) = d$. Since $f(a) < d < f(b)$, $c \neq a$, $c \neq b$, so $c \in (a, b)$.■

Note: Read on your own the material on arcwise-connectedness. Please note the discussion in the Corrections handout.

Section 2.10: Read this on your own.

Section 2.11: Continuity of Correspondences in $\mathbf{E}^n$

Definition 5 A *correspondence* $\Psi : X \to 2^Y$ from X to Y is a function from X to 2^Y, that is, $\Psi(x) \subseteq Y$ for every $x \in X$.

Examples:

1. Let $u : \mathbf{R}^n_+ \to \mathbf{R}$ be a continuous utility function, $y > 0$ and $p \in \mathbf{R}^n_{++}$, that is, $p_i > 0$ for each i. Define $\Psi : \mathbf{R}^n_{++} \times \mathbf{R}_{++} \to 2^{\mathbf{R}^n_+}$ by

$$\begin{aligned} \Psi(p,y) \quad = \quad & \arg\max u(x) \\ & \text{s.t. } x \geq 0 \\ & \qquad p \cdot x \leq y \end{aligned}$$

   Ψ is the *demand correspondence* associated with the utility function u; typically $\Psi(p,y)$ is multi-valued.[1]

2. Let $f : X \to Y$ be a function. Define $\Psi : X \to 2^Y$ by $\Psi(x) = \{f(x)\}$ for each $x \in X$. That is, we can consider a function to be the special case of a correspondence that is single-valued for each x.

Remark 6 See Item 1 on the Corrections handout. de la Fuente gives two definitions of a correspondence on page 23 that are not equivalent. The first agrees with the definition we just gave, while the second requires that for all $x \in X$, $\Psi(x) \neq \emptyset$. In asserting the equivalence of the two definitions, he seems to believe, erroneously, that $\emptyset \notin 2^Y$. In the literature, you might find the term correspondence defined in both ways, so you should check what any given author means by the term. In these notes, we do *not* impose the requirement that $\Psi(x) \neq \emptyset$. If $\Psi(x) \neq \emptyset$ for all x, we will say that Ψ is *nonempty-valued.*

We want to talk about continuity of correspondences in a way analogous to continuity of functions. We will discuss three main notions of continuity for correspondences, each of which can be motivated by thinking about what continuity means for a function $f : \mathbf{R}^n \to \mathbf{R}$.

One way a function may be discontinuous at a point x_0 is that it "jumps downward at the limit:"

$$\exists x_n \to x_0 \text{ s.t. } f(x_0) < \liminf f(x_n)$$

It could also "jump upward at the limit:"

$$\exists x_n \to x_0 \text{ s.t. } f(x_0) > \limsup f(x_n)$$

[1]The notation "arg max " here stands for the set of solutions to the given maximization problem (the argument that maximizes the given function over the given constraint set). Thus here, setting $B(p,y) = \{x \in \mathbf{R}^n : x \geq 0, p \cdot x \leq y\}$, $\Psi(p,y) = \{x^* \in B(p,y) : u(x^*) \geq u(x)\ \forall x \in B(p,y)\}$.

In either case, it doesn't matter whether the sequence x_n approaches x_0 from the left or the right (or both). See Figures 2 and 3.

What should it mean for a *set* to "jump down" at the limit x_0? It should mean the set suddenly gets smaller – it "implodes in the limit" – that is, there is a sequence $x_n \to x_0$ and points $y_n \in \Psi(x_n)$ that are far from every point of $\Psi(x_0)$ as $n \to \infty$. (See Figure 4.)

Similarly, what should it mean for a set to "jump up" at the limit? This should mean that that the set suddenly gets bigger – it "explodes in the limit" – that is, there is a point y in $\Psi(x_0)$ and a sequence $x_n \to x_0$ such that y is far from every point of $\Psi(x_n)$ as $n \to \infty$.[2] (See Figure 5.)

The first two notions of continuity below formalize this intuition.[3]

Definition 7 Let $X \subseteq \mathbf{E}^n$, $Y \subseteq \mathbf{E}^m$, and $\Psi : X \to 2^Y$.

- Ψ is *upper hemicontinuous (uhc)* at $x_0 \in X$ if, for every open set $V \supseteq \Psi(x_0)$, there is an open set U with $x_0 \in U$ such that

$$\Psi(x) \subseteq V \text{ for every } x \in U \cap X$$

- Ψ is *lower hemicontinuous (lhc)* at $x_0 \in X$ if, for every open set V such that $\Psi(x_0) \cap V \neq \emptyset$, there is an open set U with $x_0 \in U$ such that

$$\Psi(x) \cap V \neq \emptyset \text{ for every } x \in U \cap X$$

- Ψ is *continuous* at $x_0 \in X$ if it is both uhc and lhc at x_0.
- Ψ is *upper hemicontinuous* (respectively *lower hemicontinuous, continuous*) if it is uhc (respectively lhc, continuous) at every $x \in X$.

Upper hemicontinuity reflects the requirement that Ψ doesn't "implode in the limit" at x_0; lower hemicontinuity reflects the requirement that Ψ doesn't "explode in the limit" at x_0. See Figures 4 and 5 for an illustration.

Note that the definition of lower hemicontinuity does not just replace $\Psi(x_0) \subseteq V$ in the definition of upper hemicontinuity with $V \subseteq \Psi(x_0)$; indeed, we will be very interested in

[2] In de la Fuente, this intuition is reversed. He uses the terms "explode" and "implode," but not "at the limit." For him, a set explodes if it suddenly gets bigger, which agrees with our use; however, instead of looking at whether the set explodes at the limit x_0, he looks instead at whether the set explodes as you move slightly away from the limit x_0, which is equivalent to *imploding* at the limit. You may see this alternative intuition in other places as well. Figure out whether one or the other helps you to understand and remember the concepts and go with that.

[3] de la Fuente defines correspondences only with domain equalling a Euclidean space. In various applications we will be interested in correspondences defined on subsets of Euclidean space, so we modify the definitions in de la Fuente to allow for this.

correspondences in which $\Psi(x)$ has empty interior, so there will often be no open sets V such that $V \subseteq \Psi(x_0)$.

Notice also that upper and lower hemicontinuity are not nested: a correspondence can be upper hemicontinuous but not lower hemicontinuous, or lower hemicontinuous but not upper hemicontinuous.

An alternative notion of continuity looks instead at properties of the graph of the correspondence. The *graph* of a correspondence $\Psi : X \to 2^Y$ is the set

$$\text{graph } \Psi = \{(x, y) \in X \times Y : y \in \Psi(x)\}$$

Recall that a function $f : \mathbf{R}^n \to \mathbf{R}$ is continuous if and only if whenever $x_n \to x$, $f(x_n) \to f(x)$. We can translate this into a statement about its graph. Suppose $\{(x_n, y_n)\} \subseteq$ graph f and $(x_n, y_n) \to (x, y)$. Since f is a function, $(x_n, y_n) \in$ graph $f \iff y_n = f(x_n)$. Then if f is continuous, $y = \lim y_n = \lim f(x_n) = f(x)$, that is, $(x, y) \in$ graph f. So if f is continuous, each convergent sequence $\{(x_n, y_n)\}$ in graph f converges to a point (x, y) in graph f, that is, graph f is closed.

This observation suggests the third main notion of continuity for correspondences.

Definition 8 Let $X \subseteq \mathbf{E}^n$, $Y \subseteq \mathbf{E}^m$. A correspondence $\Psi : X \to 2^Y$ has *closed graph* if its graph is a closed subset of $X \times Y$, that is, if for any sequences $\{x_n\} \subseteq X$ and $\{y_n\} \subseteq Y$ such that $x_n \to x \in X$, $y_n \to y \in Y$ and $y_n \in \Psi(x_n)$ for each n, then $y \in \Psi(x)$.

Example: Consider the correspondence

$$\Psi(x) = \begin{cases} \left\{\frac{1}{x}\right\} & \text{if } x \in (0, 1] \\ \{0\} & \text{if } x = 0 \end{cases}$$

See Figure 6. Let $V = (-0.1, 0.1)$. Then $\Psi(0) = \{0\} \subset V$, but no matter how close x is to 0,

$$\Psi(x) = \left\{\frac{1}{x}\right\} \not\subseteq V$$

so Ψ is not uhc at 0. However, note that Ψ has closed graph.

Example: Consider the correspondence

$$\Psi(x) = \begin{cases} \left\{\frac{1}{x}\right\} & \text{if } x \in (0, 1] \\ \mathbf{R}_+ & \text{if } x = 0 \end{cases}$$

See Figure 7. $\Psi(0) = [0, \infty)$, and $\Psi(x) \subseteq \Psi(0)$ for every $x \in [0, 1]$. So if $V \supseteq \Psi(0)$ then $V \supseteq \Psi(x)$ for all x. Thus, Ψ is uhc, and has closed graph.

For a function, upper hemicontinuity and continuity coincide; similarly, lower hemicontinuity and continuity coincide.

Theorem 9 *Let $X \subseteq \mathbf{E}^n$, $Y \subseteq \mathbf{E}^m$ and $f : X \to Y$. Let $\Psi : X \to 2^Y$ be defined by $\Psi(x) = \{f(x)\}$ for all $x \in X$. Then Ψ is uhc if and only if f is continuous.*

Proof: Suppose Ψ is uhc. We consider the metric spaces (X, d) and (Y, d), where d is the Euclidean metric. Fix V open in Y. Then

$$\begin{aligned} f^{-1}(V) &= \{x \in X : f(x) \in V\} \\ &= \{x \in X : \Psi(x) \subseteq V\} \end{aligned}$$

Thus, f is continuous if and only if $f^{-1}(V)$ is open in X for each open V in Y, if and only if $\{x \in X : \Psi(x) \subseteq V\}$ is open in X for each open V in Y, if and only if Ψ is uhc (as an exercise, think through why this last equivalence holds). ■

For a general correspondence, these notions are not nested: a closed graph correspondence need not be uhc, as the first example above illustrates, and conversely an uhc correspondence need not have closed graph, or even have closed values.

Definition 10 A correspondence $\Psi : X \to 2^Y$ is called *closed-valued* if $\Psi(x)$ is a closed subset of X for all x; Ψ is called *compact-valued* if $\Psi(x)$ is compact for all x.

For closed-valued correspondences these concepts can be more tightly connected. A closed-valued and upper hemicontinuous correspondence must have closed graph. For a closed-valued correspondence with a compact range, upper hemicontinuity is equivalent to closed graph.

Theorem 11 (Not in de la Fuente) *Suppose $X \subseteq \mathbf{E}^n$ and $Y \subseteq \mathbf{E}^m$, and $\Psi : X \to 2^Y$.*

(i) If Ψ is closed-valued and uhc, then Ψ has closed graph.

(ii) If Ψ has closed graph and there is an open set W with $x_0 \in W$ and a compact set Z such that $x \in W \cap X \Rightarrow \Psi(x) \subseteq Z$, then Ψ is uhc at x_0.

(iii) If Y is compact, then Ψ has closed graph $\iff$ Ψ is closed-valued and uhc.

Proof: (i) Suppose Ψ is closed-valued and uhc. If Ψ does not have closed graph, we can find a sequence $(x_n, y_n) \to (x_0, y_0)$, where (x_n, y_n) lies in the graph of Ψ (so $y_n \in \Psi(x_n)$) but (x_0, y_0) does not lie in the graph of Ψ (so $y_0 \notin \Psi(x_0)$). Since Ψ is closed-valued, $\Psi(x_0)$ is closed. Since $y_0 \notin \Psi(x_0)$, there is some $\varepsilon > 0$ such that

$$\Psi(x_0) \cap B_{2\varepsilon}(y_0) = \emptyset$$

so

$$\Psi(x_0) \subseteq \mathbf{E}^n \setminus B_\varepsilon[y_0]$$

Let $V = \mathbf{E}^m \setminus B_\varepsilon[y_0]$. Then V is open, and $\Psi(x_0) \subseteq V$. Since Ψ is uhc, there is an open set U with $x_0 \in U$ such that

$$x \in U \cap X \Rightarrow \Psi(x) \subseteq V$$

Since $(x_n, y_n) \to (x_0, y_0)$, $x_n \in U$ for n sufficiently large, so

$$y_n \in \Psi(x_n) \subseteq V$$

Thus for n sufficiently large, $|y_n - y_0| \geq \varepsilon$, which implies that $y_n \not\to y_0$, and $(x_n, y_n) \not\to (x_0, y_0)$, a contradiction. Thus Ψ is closed-graph.

(ii) Now, suppose Ψ has closed graph and there is an open set W with $x_0 \in W$ and a compact set Z such that

$$x \in W \cap X \Rightarrow \Psi(x) \subseteq Z$$

Since Ψ has closed graph, it is closed-valued. Let V be any open set such that $V \supseteq \Psi(x_0)$. We need to show there exists an open set U with $x_0 \in U$ such that

$$x \in U \cap X \Rightarrow \Psi(x) \subseteq V$$

If not, we can find a sequence $x_n \to x_0$ and $y_n \in \Psi(x_n)$ such that $y_n \notin V$. Since $x_n \to x_0$, $x_n \in W \cap X$ for all n sufficiently, and thus $\Psi(x_n) \subseteq Z$ for n sufficiently large. Since Z is compact, we can find a convergent subsequence $y_{n_k} \to y'$. Then $(x_{n_k}, y_{n_k}) \to (x_0, y')$. Since Ψ has closed graph, $y' \in \Psi(x_0)$, so $y' \in V$. Since V is open, $y_{n_k} \in V$ for all k sufficiently large, a contradiction. Thus, Ψ is uhc at x_0.

(iii) Follows from (i) and (ii). ■

Upper and lower hemicontinuity can be given sequential characterizations that are useful in applications.

Theorem 12 (Thm. 11.2) *Suppose $X \subseteq \mathbf{E}^n$ and $Y \subseteq \mathbf{E}^m$. A compact-valued correspondence $\Psi : X \to 2^Y$ is uhc at $x_0 \in X$ if and only if, for every sequence $\{x_n\} \subseteq X$ with $x_n \to x_0$ and every sequence $\{y_n\}$ such that $y_n \in \Psi(x_n)$ for each n, there is a convergent subsequence $\{y_{n_k}\}$ such that $\lim y_{n_k} \in \Psi(x_0)$.*

Proof: See de la Fuente. ■

Note that this characterization of upper hemicontinuity requires the correspondence to have compact values.

Theorem 13 (Thm. 11.3) *A correspondence $\Psi : X \to 2^Y$ is lhc at $x_0 \in X$ if and only if, for every sequence $\{x_n\} \subseteq X$ with $x_n \to x_0$, and every $y_0 \in \Psi(x_0)$, there exists a companion sequence $\{y_n\}$ with $y_n \in \Psi(x_n)$ for each n such that $y_n \to y_0$.*

Proof: See de la Fuente. ■

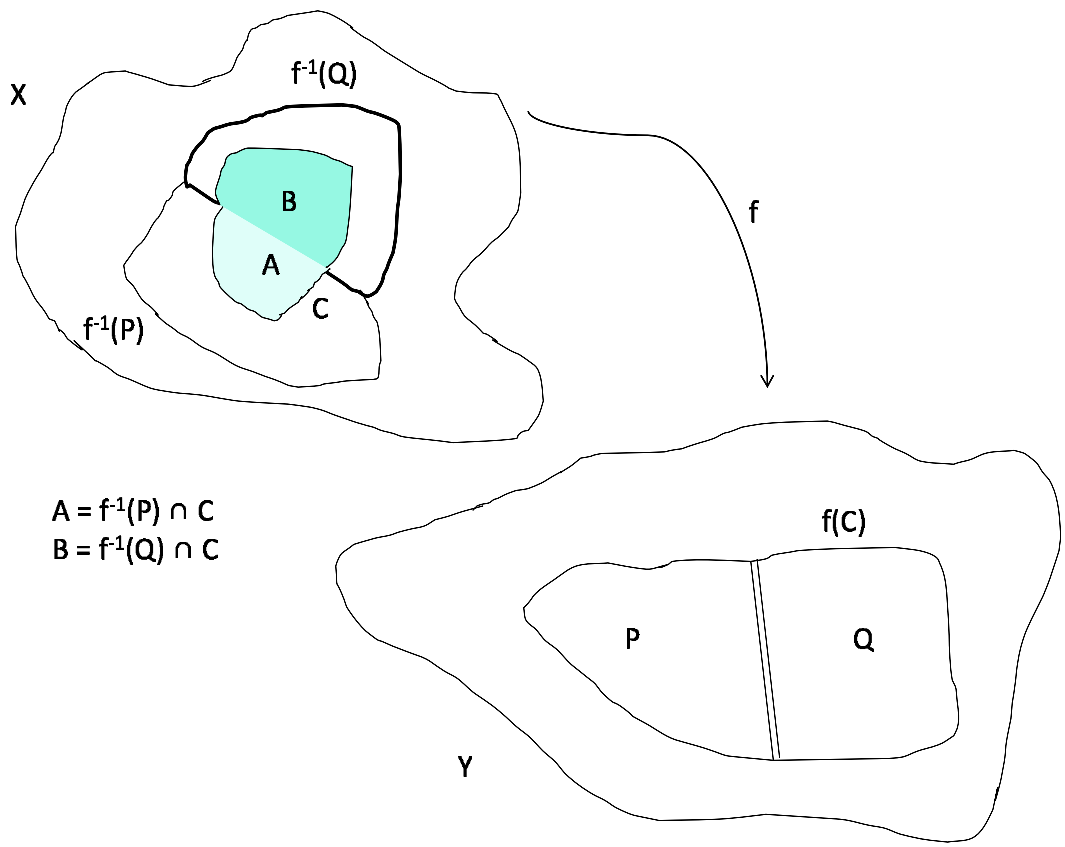

Figure 1: Continuity preserves connectedness: if f is continuous and C is connected, then $f(C)$ is connected. The picture gives the idea behind the proof.

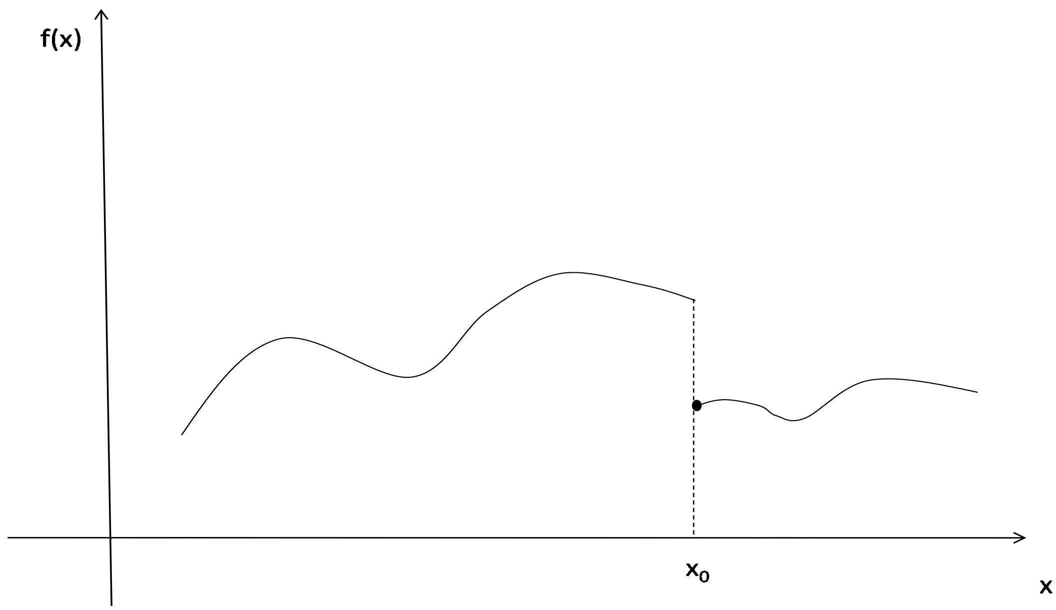

Figure 2: The function f "jumps down" in the limit at x_0.

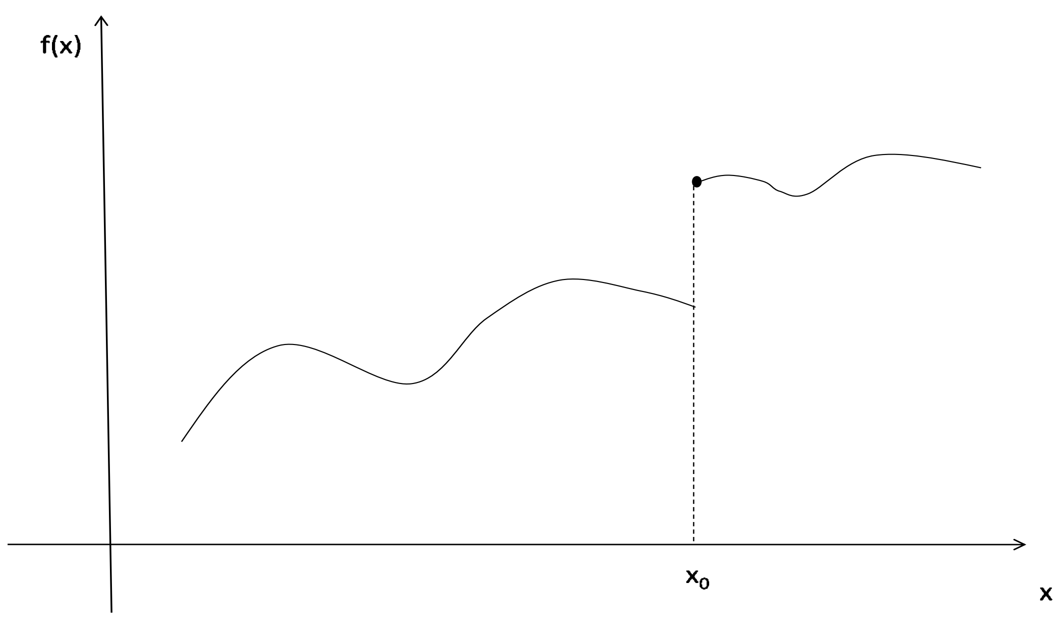

Figure 3: The function f "jumps up" in the limit at x_0.

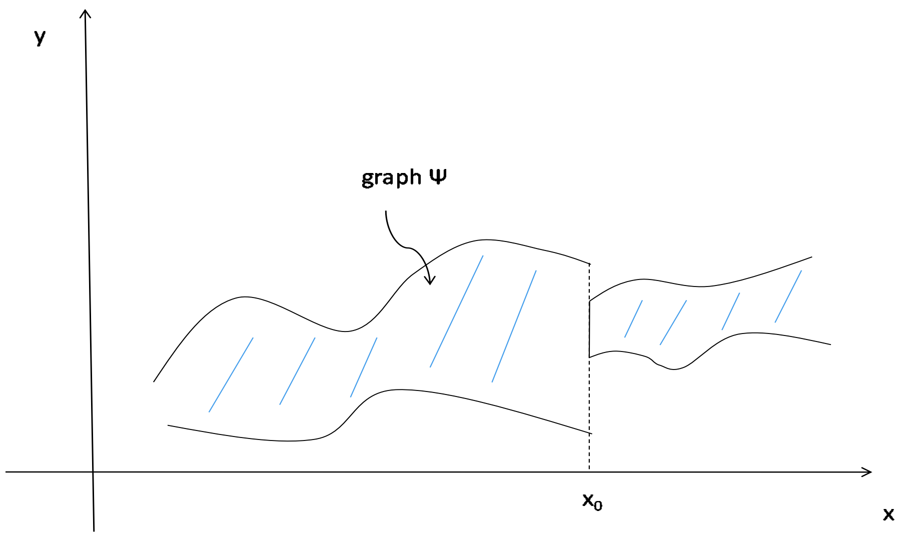

Figure 4: The correspondence Ψ "implodes in the limit" at x_0. Ψ is not upper hemicontinuous at x_0.

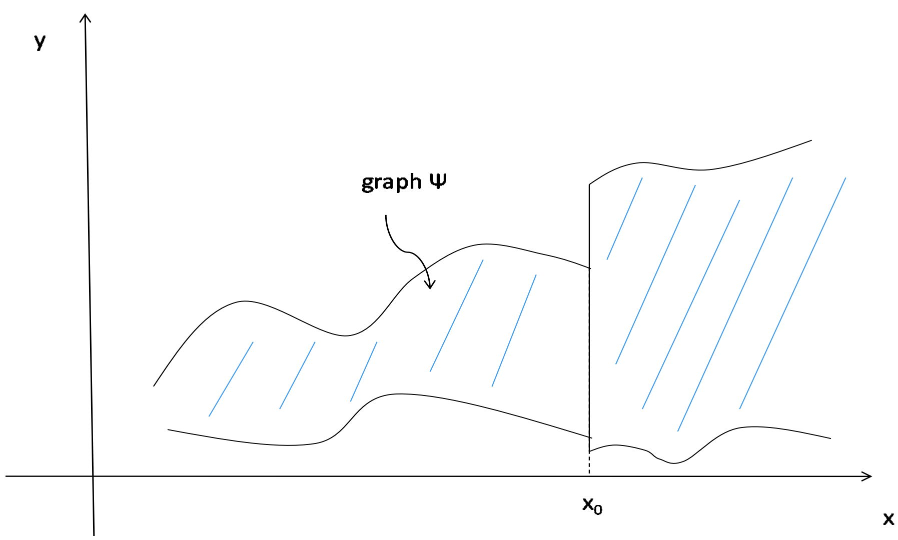

Figure 5: The correspondence Ψ "explodes in the limit" at x_0. Ψ is not lower hemicontinuous at x_0.

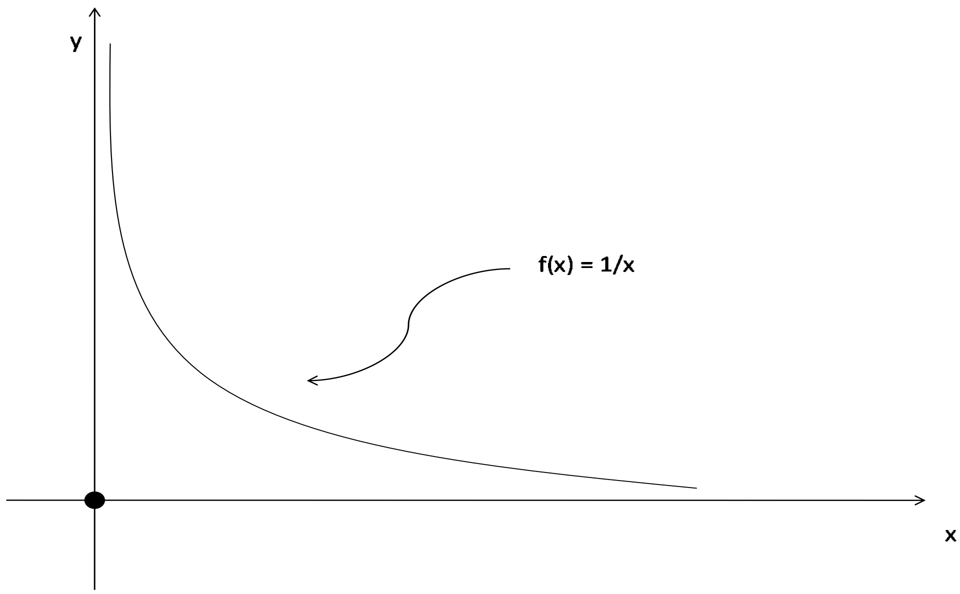

Figure 6: A correspondence that has closed graph but is not upper hemicontinuous.

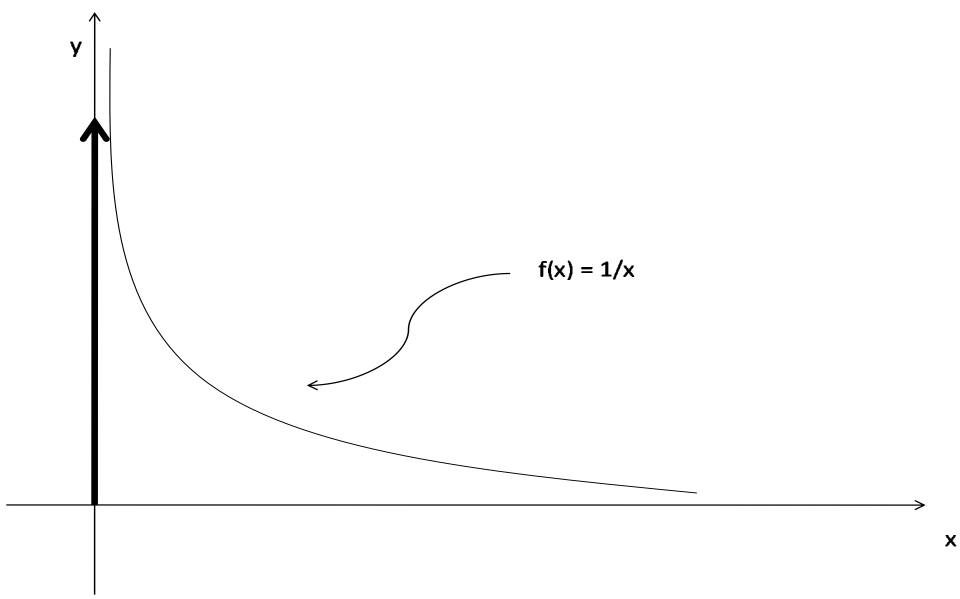

Figure 7: By changing the value of 0, now the correspondence is upper hemicontinuous, and also has closed graph.

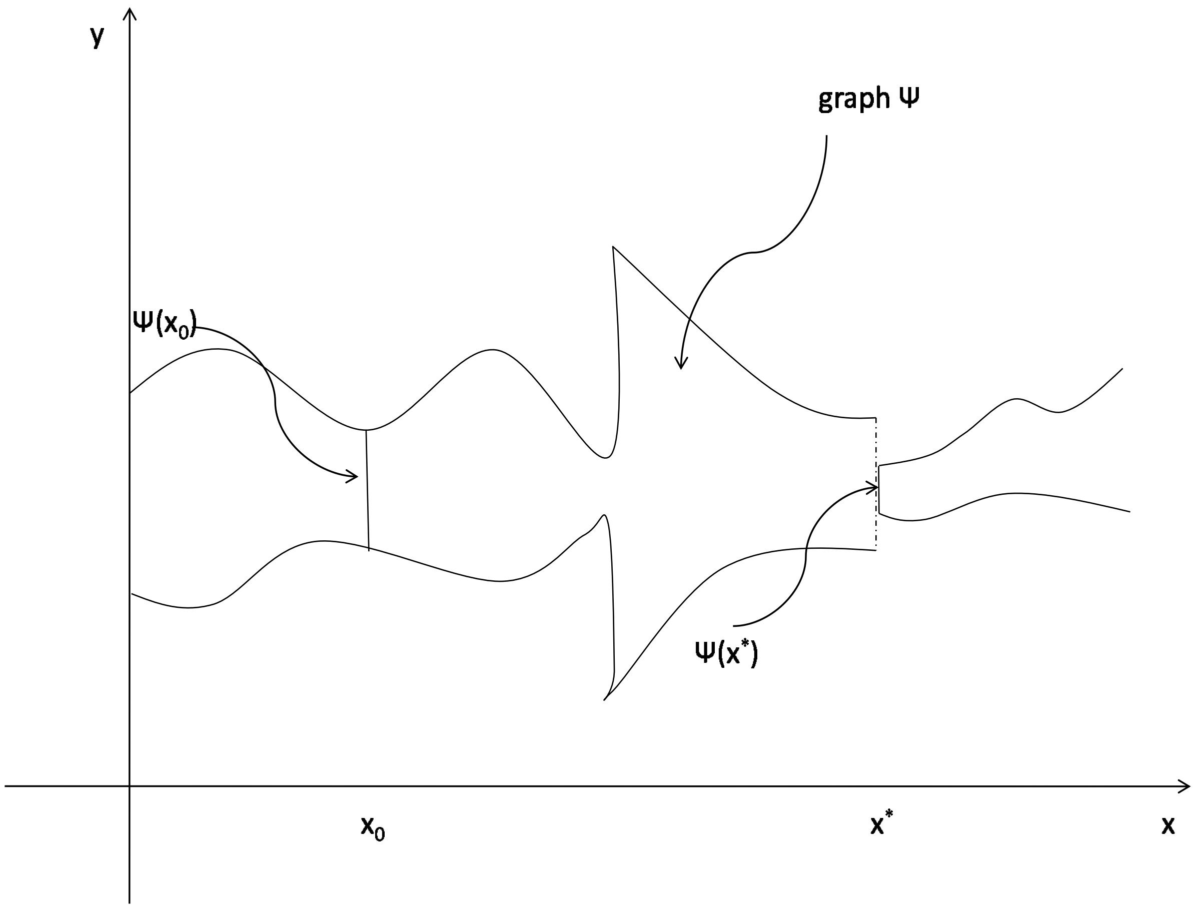

Figure 8: The correspondence Ψ is neither upper hemicontinuous nor lower hemicontinuous (why?).