Economics 204
Fall 2010
Problem Set 1 Solutions

1. Some practice with set theory:

(a) Determine the truth of the following statements. Prove them if true, if not provide counter-examples.[1]

i. $A \setminus B = C \implies A = B \cup C$

ii. $A = B \cup C \implies A \setminus B = C$

Solution.
i. No, $A \setminus B = C$ only implies that $A \subset B \cup C$.
ii. No, $A = B \cup C$ only implies $A \setminus B \subset C$.

For counterexample to (i.) consider $A = \{1, 2\}$ and $B = \{2, 3\}$ hence $C = \{1\}$. But $A = \{1, 2\} \neq \{1, 2, 3\} = C \cup B$.
For counterexample to (ii.) consider $B = C = \{1\}$ hence $A = \{1\}$ but $A \setminus B = \{\emptyset\} \neq \{1\} = C$.

(b) Establish the relationship between sets X and Y ($X \subset Y$, $X \supset Y$, $X = Y$, or none of the above), if [2]

i. $X = A \cup (B \setminus C)$, $Y = (A \cup B) \setminus (A \cup C)$;

ii. $X = (A \cap B) \setminus C$, $Y = (A \setminus C) \cap (B \setminus C)$;

iii. $X = A \setminus (B \cup C)$, $Y = (A \setminus B) \cup (A \setminus C)$.

Solution.
i. $X \supset Y$, ii. $X = Y$, iii. $X \subset Y$.

2. Let $f : A \to B$ and let B_1 and B_2 be subsets of B.

(a) Show that f^{-1} preserves intersections and differences of sets

$$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$$

$$f^{-1}(B_1 \setminus B_2) = f^{-1}(B_1) \setminus f^{-1}(B_2)$$

Solution. Lets prove this two statements through two-way set inclusion. In other words, we will first show that $f^{-1}(B_1 \cap B_2) \supseteq f^{-1}(B_1) \cap f^{-1}(B_2)$

[1] Note that $A \setminus B$ means the set difference between A and B, denoted by $A \sim B$ in de la Fuente. Thus, to clarify, $A \setminus B = \{x \in A \mid x \notin B\}$.

[2] *None of the above* would mean that sets X and Y are not comparable under set inclusion.

and then $f^{-1}(B_1 \cap B_2) \subseteq f^{-1}(B_1) \cap f^{-1}(B_2)$. We prove that f^{-1} preserves differences of sets in the same way.

So, take $b \in f^{-1}(B_1) \cap f^{-1}(B_2)$, then $f(b) \in B_1$ and $f(b) \in B_2$. This implies $f(b) \in B_1 \cap B_2$. Thus, $b \in f^{-1}(B_1 \cap B_2)$ and we have shown that

$$f^{-1}(B_1 \cap B_2) \supseteq f^{-1}(B_1) \cap f^{-1}(B_2).$$

Conversely, observe that for any sets B_1 and B_2 we must have $f^{-1}(B_1 \cap B_2) \subseteq f^{-1}(B_1)$ and $f^{-1}(B_1 \cap B_2) \subseteq f^{-1}(B_2)$. Thus, we have

$$f^{-1}(B_1 \cap B_2) \subseteq f^{-1}(B_1) \cap f^{-1}(B_2).$$

Combining these two set inclusion statements we get the result we seek.

Now, take $b \in f^{-1}(B_1 \setminus B_2)$ then $f(b) \in B_1 \setminus B_2$. So, $f(b) \notin B_2$ but $f(b) \in B_1$. So, we must have $b \in f^{-1}(B_1) \setminus f^{-1}(B_2)$ and thus

$$f^{-1}(B_1 \setminus B_2) \subseteq f^{-1}(B_1) \setminus f^{-1}(B_2).$$

Conversely, take $b \in f^{-1}(B_1) \setminus f^{-1}(B_2)$, then $b \notin f^{-1}(B_2)$. So, $f(b) \notin B_2$ but $f(b) \in B_1$. That is $f(b) \in B_1 \setminus B_2$. Therefore, $b \in f^{-1}(B_1 \setminus B_2)$ and we have

$$f^{-1}(B_1 \setminus B_2) \supseteq f^{-1}(B_1) \setminus f^{-1}(B_2).$$

Again, as before, by combining these two set inclusion statements we get the result we seek.

(b) Is the same true about f? Does it preserve intersections and differences of sets? For the case(s) it does not, please provide examples (you do not need to prove those facts rigorously).

Solution. No, it is not true both for intersections and differences of sets, unless f is injective. Take intersection of sets first. Let A_1 and A_2 be subsets of A. Clearly, the expression $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ is not always true. Consider, for instance, $f(x) = \sin x$ and $A_1 = [0, \frac{3\pi}{4}]$, $A_2 = [\frac{\pi}{2}, \frac{3\pi}{2}]$, then we have $f(A_1) = [0, 1]$, $f(A_2) = [-1, 1]$, thus,

$$f(A_1 \cap A_2) = f\left(\left[\frac{\pi}{2}, \frac{3\pi}{4}\right]\right) \neq f(A_1) \cap f(A_2) = [0, 1].$$

For a general f the following set inclusion is true: $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$.

Now consider set differences. Let $f(x) = x^2$, $A_1 = (-\infty, +\infty)$, $A_2 = [0, +\infty)$, then

$$f(A_1 \setminus A_2) = f((-\infty, 0)) = (0, +\infty).$$

Therefore, $f(A_1 \setminus A_2) \supset f(A_1) \setminus f(A_2)$ and we definitely do not get an equality because $f(A_1) \setminus f(A_2) = \emptyset$. When f is not injective one can only ascertain that $f(A_1 \setminus A_2) \supset f(A_1) \setminus f(A_2)$.

(c) What about inclusion and unions of sets (both for f and f^{-1})? Just state the result, no explanation is necessary.

Solution. Yes, it is true. Inclusion and unions of sets are preserved both under f and inverse mapping f^{-1}, i.e. we have

$$B_1 \subset B_2 \implies f^{-1}(B_1) \subset f^{-1}(B_2)$$
$$f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$$

and

$$A_1 \subset A_2 \implies f(A_1) \subset f(A_2)$$
$$f(A_1 \cup A_2) = f(A_1) \cup f(A_2).$$

3. Using induction prove

(a) Imagine that the only money in the world are three and five cents coins. Prove that you can pay (without change!) any sum greater then seven cents.

Solution. First, consider the case where n = 8 (the base case): it is clear that 8 cents can be paid by 5 and 3 cents, 8=5+3. Now suppose that the statement holds for some n (the inductive hypothesis). We want to show that it holds for $n+1$ as well (the inductive step). Consider two cases: in n cent sum there is at least one 5 cent coin or none. In the former case, replace the 5 cent coin by three 3 cent ones. In the latter case, n cents are represented by 3 cent coins only, and it is clear that there are no less then three of them. Therefore, replace three 3 cent coins by two 5 cent coins.

Thus, we have show that if n cents can be paid by five and three cent coins only, so can $n+1$ cents. We are done.

(b) Let $n > 1$ and $x > -1$, prove that $(1+x)^n \geq 1+nx$. When is the inequality sharp?

Solution. The claim is trivially true for any n if $x = 0$. So, without any loss of generality take $x \neq 0$. First, consider the case where n = 2 (the base case): it is clear that inequality is true $(1+x)^2 = 1+2x+x^2 > 1+2x$ because $x^2 > 0$, $x \neq 0$. Now suppose that the statement holds for some n (the inductive hypothesis), i.e. $(1+x)^n > 1+nx$ for $n \geq 2$, $n \in \mathbf{N}$. We want to show that it holds for $n+1$ as well (the inductive step). Lets multiply both sides of the inequality in the inductive step by $1+x$ (which is strictly positive by our assumption), we obtain

$$(1+x)^n(1+x) > (1+nx)(1+x)$$
$$(1+x)^{n+1} > (1+(n+1)x) + nx^2$$

Since $nx^2 > 0$ we obtain the result we seek.

Clearly, inequality is sharp if and only if $x = 0$.

4. Let $f : A \to B$ be a surjective function. Let us define a binary relation R on A by $a_0 \, R \, a_1$ if $f(a_0) = f(a_1)$.

   (a) Show this is an equivalence relation.

   Solution. To show that R is an equivalence relation on A, we need to show that it is reflexive, symmetric and transitive. All these properties of R follow naturally from the corresponding properties for equality, i.e.

   reflexivity Clearly, $\forall a_0 \in A,\ a_0 R a_0$ because $a_0 \, R \, a_0 \iff f(a_0) = f(a_0)$.

   symmetry We need to show that $\forall a_0,\ a_1 \in A,\ a_0 \, R \, a_1 \iff a_1 \, R \, a_0$. This is true because equality is symmetric: $a_0 \, R \, a_1 \implies (f(a_0) = f(a_1)) \implies (f(a_1) = f(a_0)) \implies a_1 \, R \, a_0$

   transitivity We need to show that $\forall a_0,\ a_1,\ a_2 \in A,\ (a_0 \, R \, a_1 \wedge a_1 \, R \, a_2) \implies a_0 \, R \, a_2$. Again, this is true because $a_0 \, R \, a_1 \implies f(a_0) = f(a_1)$ and $a_1 \, R \, a_2 \implies f(a_1) = f(a_2)$. Since $(f(a_0) = f(a_1) = f(a_2)) \implies (f(a_0) = f(a_2))$, we have that $a_0 \, R \, a_2$.

   (b) Let A^* be the set of equivalence classes. Show there is a bijective correspondence of A^* with B.

   Solution. Let $A^* = \{[a] \,|\, a \in A\}$ be the equivalence classes that R induces on A. Given a function $f : A \to B$ we can define a function f^* that will map the equivalence classes into elements of range of f, i.e. $f^* : A^* \to B$. We define f^* in the following way: $f^*([a]) = f(a)$ and claim that f is a bijection.

   To show that we need to demonstrate that f is surjective and one-to-one. Surjection is immediate, since given $b \in B$, there an $a \in A$ such that $f(a) = b$ because f is surjective. Because $f^*([a]) = b$, f^* is clearly surjective. To see that f^* is one-to-one, suppose, that $f^*([a_0]) = f^*([a_1])$. By the definition of f^* we get $f(a_0) = f(a_1)$ which means that $a_0 \, R \, a_1$. Therefore, a_0 and a_1 are in the same equivalence class, $[a_0] = [a_1]$, thus, f^* must be one-to-one.

5. Let A be a countable set and $f : A \to B$. Show that B is at most countable if $B = f(A)$.

   Solution. Since A is countable we can find a bijection $a : \mathbf{N} \to A$. For each n set $a_n = a(n)$. Then for each $b \in B$ define $m(b) = \min\{n \in \mathbf{N} : f(a_n) = b\}$ and let $\mathbf{N}_f = m(B) \subseteq \mathbf{N}$. Then $f \circ a : m(B) \to B$ is a bijection: if $n \neq n'$ then $a_n \neq a_{n'}$ and $n, n' \in m(B) \Rightarrow f(a_n) \neq f(a_{n'})$. Since $B = f(A)$, $f \circ a$ is clearly onto. If $m(B)$ is finite we're done, so suppose $m(B)$ is infinite. We must show it is countable. Since $m(B) \subseteq \mathbf{N}$,

$$\begin{aligned}
\text{set } n_0 &= \min m(B) \\
\text{set } n_1 &= \min m(B) \setminus \{n_0\} \\
\text{set } n_2 &= \min m(B) \setminus \{n_0, n_1\} \\
&\vdots \\
\text{set } n_k &= \min m(B) \setminus \{n_0, n_1, \ldots, n_{k-1}\}
\end{aligned}$$

Since $m(B)$ is infinite, $n_1 < n_2 < \ldots$ and n_k is well-defined for each k. For each $n \in m(B)$ there exists k such that $n = n_k$. Thus $m(B)$ is countable.

It is important to emphasize that there is a general result that underlines our proof above: if X is countable and $S \subseteq X$ is nonempty, then S is either finite or countable. Observe that above we showed this for $\mathbf{N}$. To see this, suppose S is an infinite subset of X (as above, if S is finite we're done). Let $g : \mathbf{N} \to X$ be a bijection. Since S is infinite, $\{n \in \mathbf{N} : g(n) \in S\}$ is infinite. Define $h : \mathbf{N} \to S$ as follows:

$$\begin{aligned}
\text{set } n_0 &= \min\{n \in \mathbf{N} : g(n) \in S\} \text{ and } h(0) = g(n_0) \\
\text{set } n_1 &= \min\{n \in \mathbf{N} \setminus \{n_0\} : g(n) \in S\} \text{ and } h(1) = g(n_1) \\
\text{set } n_2 &= \min\{n \in \mathbf{N} \setminus \{n_0, n_1\} : g(n) \in S\} \text{ and } h(2) = g(n_2) \\
&\vdots \\
\text{set } n_k &= \min\{n \in \mathbf{N} \setminus \{n_0, n_1, \ldots, n_{k-1}\} : g(n) \in S\} \text{ and } h(k) = g(n_k)
\end{aligned}$$

Since S is infinite, $n_1 < n_2 < \ldots$ and n_k is well-defined for each k. Thus $h : \mathbf{N} \to S$. Since g is 1-1, h is 1-1. Since g is onto and $S \subseteq X$, for each $s \in S$ there exists $n \in \mathbf{N}$ such that $g(n) = s$, so $\exists k$ such that $n_k = n$ and $h(k) = n$. So h is onto, which shows that S is countable.

From this it follows that if there exists a 1-1 function $r : S \to \mathbf{N}$ then S is either finite or countable (because $r : S \to r(S)$ is a bijection and $r(S) \subseteq \mathbf{N}$ must be either finite or countable).

Equivalently, if there exists a surjection $f : \mathbf{N} \to S$ then S is either finite or countable. This follows because for each $s \in S$, $f^{-1}(s) = \{n \in \mathbf{N} : f(n) = s\} \neq \emptyset$, so set $r(s) = \min\{n \in \mathbf{N} : f(n) = s\}$. Then $r : S \to \mathbf{N}$ is 1-1, so by the above argument S is either finite or countable.

Finally, there is another way to establish the this fact using results from the previous exercise part (b). As above, without any loss of generality take $A = \mathbf{N}$. Because A^* defined as in the previous exercise is a partition of A, there is a bijection g from A^* into subsets of A which we define by choosing an element in each set $[a]$. Now, since $f^* : A^* \to B$ is a bijection, $(f^*)^{-1} \circ g$ gives a bijection from B to a subset of A.

6. Let X and Y be non-empty sets of $\mathbf{R}$, such that

   - for any $x \in X$ and any $y \in Y$ we have $x \leq y$
   - for any $\epsilon > 0$ there is $x_\epsilon \in X$ and $y_\epsilon \in Y$, such that $y_\epsilon - x_\epsilon < \epsilon$

   Show that $\sup X = \inf Y$.

   Solution. Firstly note that X is bounded above and Y is bounded below so $\sup X$ and $\inf Y$ are just some real numbers. Before proceeding with the proof lets state and prove a claim that will be useful for us later:

   $$x \leq y \;\; \forall x \in X, \forall y \in Y \implies \sup X \leq \inf Y$$

   Fix $y \in Y$. Since $x \leq y \quad \forall x \in X \quad y$ is an upper bound for X. Thus, by definition of sup we have $\sup X \leq y$. Now, since our choice of $y \in Y$ was arbitrary, we have that $\sup X \leq y \quad \forall y \in Y$. Thus, $\sup X$ is a lower bound for Y. This implies that $\sup X \leq \inf Y$ by definition of infimum.

   Lets prove the claim. Assume, to contradiction, that $\sup X < \inf Y$ (the other case is impossibly by our result above). So, let $\epsilon = \inf Y - \sup X$ and observe that under our hypothesis $\epsilon > 0$. Given $\frac{\epsilon}{2}$ lets find such $x_\epsilon \in X$ and $y_\epsilon \in Y$ so that $y_\epsilon - x_\epsilon < \frac{\epsilon}{2}$. By definition of *inf* we must have $y_\epsilon \geq \inf Y$ and $x_\epsilon \leq \sup X$. Now, if we can show that $\sup X < x_\epsilon$ we would get a contradiction and would be done. So note

   $$\sup X = \inf Y - \epsilon \leq \frac{\epsilon}{2} + x_\epsilon - \epsilon \implies \sup X \leq x_\epsilon - \frac{\epsilon}{2}$$

   Thus, $\sup X < x_\epsilon$, because $\epsilon > 0$.

7. Determine which of the following is a metric on $\mathbb{R}$?

   (a) $d(x, y) = |x - 2y|$

   (b) $d(x, y) = \frac{|x-y|}{1+|x-y|}$

   Solution. Recall the defining properties of a metric:
   1. $d(x, y) \geq 0,\ d(x, y) = 0 \iff x = y \;\; \forall x, y \in \mathbb{R}$
   2. $d(x, y) = d(y, x) \;\; \forall x, y \in \mathbb{R}$
   3. $d(x, y) + d(y, z) \geq d(x, z) \;\; \forall x,\ y,\ z \in \mathbb{R}$.

   The function in (b) is a metric, however, the one in (a) is not because of the violation of one of the property (1) above. $d(x, x) = |x| \neq 0$, unless $x = 0$.

   The relatively straightforward verification of properties (1) and (2) for (b) is omitted. We verify the triangle inequality. Note that

   $$\frac{|x-y|}{1+|x-y|} = \frac{1+|x-y|}{1+|x-y|} - \frac{1}{1+|x-y|} = 1 - \frac{1}{1+|x-y|}$$

To simplify calculations a bit, lets denote $a = |x - y|$, $b = |y - z|$, $c = |x - z|$ and note that triangle inequality for absolute value in $\mathbf{R}$ implies that $a + b \geq c$. Then

$$1 - \frac{1}{1+a} + 1 - \frac{1}{1+b} \geq 1 - \frac{1}{1+c} \iff$$
$$\frac{1}{1+a} + \frac{1}{1+b} \leq 1 + \frac{1}{1+c} \iff$$
$$\frac{2+a+b}{1+a+b+ab} \leq \frac{2+c}{1+c} = \frac{1}{1+c} + 1$$

Lets work with left hand side of this inequality first

$$\frac{1+1+a+b}{1+a+b+ab} \leq \frac{1}{1+a+b+ab} + \frac{1+a+b+ab}{1+a+b+ab}$$
$$\leq \frac{1}{1+a+b+ab} + 1$$

Now, we have

$$a + b \geq c \implies a + b + ab \geq c \implies$$
$$\frac{1}{1+a+b+ab} \leq \frac{1}{1+c} \implies$$
$$\frac{1}{1+a+b+ab} + 1 \leq \frac{1}{1+c} + 1.$$

So, we are done.

8. Suppose that a sequence $\{x_n\}$ in a metric space has the property that exists x, such that any subsequence has in turn a further subsequence that converges to x. Prove that $\{x_n\} \to x$ and that converse is also true.

Solution.

Assume every subsequence x_{n_k} itself has a further subsequence that converges to x. We wish the show that this implies the convergence of the full sequence. Proceeding by contradiction, assume not. Then, from the definition, there must be some $\epsilon > 0$ such that $d(x_n, x) > \epsilon$ for infinitely many n. Define x_{n_k} to then be the (infinite) subsequence of x_n for which this is true. It then follows that x_{n_k} itself has no subsequence that converges to x since every term of the full sequence was chosen to be at least distance ϵ from x. Contradiction.

Lets now prove the converse. If the sequence converges to x, then clearly every subsequence converges to x. To check this let x_{n_k} be a subsequence. For any $\epsilon > 0$ convergence of x_n tells us that $\exists N(\epsilon)$ such that $n > N(\epsilon) \Rightarrow d(x_n, x) < \epsilon$. In particular, for all $n_k > N(\epsilon)$ we must have $d(x_{n_k}, x) < \epsilon$. Hence, the subsequence x_{n_k} converges to x. Now note that x_{n_k} is a subsequence of itself and because it converges to x we have found a subsequence of x_{n_k} that converges to x.