Economics 204
Fall 2010
Problem Set 1
Due Friday, July 30 in Lecture

1. Some practice with set theory:

   (a) Determine the truth of the following statements. Prove them if true, if not provide counter-examples.[1]

      i. $A \setminus B = C \implies A = B \cup C$
      ii. $A = B \cup C \implies A \setminus B = C$

   (b) Establish the relationship between sets $X$ and $Y$ ($X \subset Y$, $X \supset Y$, $X = Y$, or none of the above), if [2]

      i. $X = A \cup (B \setminus C)$, $Y = (A \cup B) \setminus (A \cup C)$;
      ii. $X = (A \cap B) \setminus C$, $Y = (A \setminus C) \cap (B \setminus C)$;
      iii. $X = A \setminus (B \cup C)$, $Y = (A \setminus B) \cup (A \setminus C)$.

2. Let $f : A \to B$ and let $B_1$ and $B_2$ be subsets of B.

   (a) Show that $f^{-1}$ preserves intersections and differences of sets

$$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$$

$$f^{-1}(B_1 \setminus B_2) = f^{-1}(B_1) \setminus f^{-1}(B_2)$$

   (b) Is the same true about $f$? Does it preserve intersections and differences of sets? For the case(s) it does not, please provide examples (you do not need to prove those facts rigorously).

   (c) What about inclusion and unions of sets (both for $f$ and $f^{-1}$)? Just state the result, no explanation is necessary.

3. Using induction prove

   (a) Imagine that the only money in the world are three and five cents coins. Prove that you can pay (without change!) any sum greater then seven cents.

   (b) Let $n > 1$ and $x > -1$, prove that $(1+x)^n \geq 1+nx$. When is the inequality sharp?

4. Let $f : A \to B$ be a surjective function. Let us define a binary relation $R$ on $A$ by $a_0 \, R \, a_1$ if $f(a_0) = f(a_1)$.

[1]Note that $A \setminus B$ means the set difference between $A$ and $B$, denoted by $A \sim B$ in de la Fuente. Thus, to clarify, $A \setminus B = \{x \in A \mid x \notin B\}$.

[2]*None of the above* would mean that sets $X$ and $Y$ are not comparable under set inclusion.

(a) Show this is an equivalence relation.

(b) Let $A^*$ be the set of equivalence classes. Show there is a bijective correspondence of $A^*$ with B.

5. Let $A$ be a countable set and $f : A \to B$. Show that $B$ is at most countable, if $B = f(A)$.

6. Let $X$ and $Y$ be non-empty sets of $\mathbb{R}$, such that

   - for any $x \in X$ and any $y \in Y$ we have $x \leq y$
   - for any $\epsilon > 0$ there is $x_\epsilon \in X$ and $y_\epsilon \in Y$, such that $y_\epsilon - x_\epsilon < \epsilon$

   Show that $\sup X = \inf Y$.

7. Determine which of the following is a metric on $\mathbb{R}$?

   (a) $d(x, y) = |x - 2y|$

   (b) $d(x, y) = \frac{|x-y|}{1+|x-y|}$

8. Suppose that a sequence $\{x_n\}$ in a metric space has the property that exists $x$, such that any subsequence has in turn a subsequence that converges to $x$. Prove that $\{x_n\} \to x$ and that converse is also true.