Economics 204
Fall 2010
Problem Set 2 Suggested Solutions

1. Determine whether the following sets are open, closed, both or neither:

   (a) $\mathbb{Z}$ in the topology on $\mathbb{R}$ induced by the usual metric;

   (b) $\{1/n \mid n \in \mathbb{N}\}$ in the topology on $\mathbb{R}$ induced by the usual metric;

   (c) $\mathbb{Q}$ in the topology on $\mathbb{R}$ induced by the usual metric;

   (d) $\{(x, y) \in \mathbb{R}^2 \mid y \geq x^2\}$;

   (e) $\{(x, y) \in \mathbb{R}^2 \mid |y| > x\}$.

   **Solution:**

   (a) Closed: since $\forall j \in \mathbb{Z}$, $(j, j+1)$ is open, so is the infinite union $\bigcup_{j=-\infty}^{\infty}(j, j+1)$, i.e. $\mathbb{R}\backslash\mathbb{Z}$ is open. Hence, $\mathbb{Z}$ is closed. On the other hand, every open ball around any $j \in \mathbb{Z}$ is clearly not contained in $\mathbb{Z}$, so $\mathbb{Z}$ is not open.

   (b) Neither: $\forall \varepsilon > 0$, $B_\varepsilon(1)$ is not contained in the set, so it is not open. Consider the sequence $\{d_m = 1/m\}$ for $m \in \mathbb{N}$. Obviously, $d_m$ is contained in the set and converges to 0 which does not belong to the set, so it is not closed either.

   (c) Neither: consider the sequence $\{\sqrt{2}/m\}$ for $m \in \mathbb{N}$. We know that $\forall m \in \mathbb{N}, \sqrt{2}/m \in \{\mathbb{R}\backslash\mathbb{Q}\}$. However, that sequence converges to $0 \in \mathbb{Q}$, i.e. $\mathbb{R}\backslash\mathbb{Q}$ is not closed and therefore $\mathbb{Q}$ is not open. On the other hand, note that $\sqrt{2}$ can be written as the infinite decimal $1.41421356\ldots$. Let $x^1 = 1, x^2 = 1.4, x^3 = 1.41, \ldots, x^m$ =the number formed by the first $m$ digits of the decimal expansion of $\sqrt{2}$. Then $x^m \to \sqrt{2} \notin \mathbb{Q}$, i.e. $\mathbb{Q}$ is not closed.

   (d) Closed: first, note that if a function $g : \mathbb{R} \to \mathbb{R}$ is continuous, then the functions $H, G : \mathbb{R}^2 \to \mathbb{R}$ defined by:

   $$G(x, y) = g(x); H(x, y) = g(y)$$

   are continuous. Then for some open $A \subseteq \mathbb{R}$ we have:

   $$\begin{aligned} G^{-1}(A) &= g^{-1}(A) \times \mathbb{R} \\ H^{-1}(A) &= \mathbb{R} \times g^{-1}(A). \end{aligned}$$

   If $g$ is continuous, $g^{-1}(A)$ is open. Since $\mathbb{R}$ is open, $G^{-1}(A)$ and $H^{-1}(A)$ are also open and, hence, $G$ and $H$ are continuous.

   We can then use this to show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = y - x^2$ is continuous. Letting $H(x, y) = y$ and $G(x, y) = x^2$, $H(x, y)$ and $G(x, y)$ are continuous on $\mathbb{R}^2$ by the argument above and hence $f = H - G$ is continuous as well. Now consider the set we are interested in:

$$\begin{aligned} & \{(x, y) \in \mathbb{R}^2 : y \geq x^2\} \\ = & \{(x, y) \in \mathbb{R}^2 : f(x, y) \geq 0\} \\ = & f^{-1}([0, \infty)) \end{aligned}$$

$f$ is continuous and $[0, \infty)$ is closed; therefore, the set under consideration must be closed as it is a continuous inverse image of a closed set.

The set is not open. We see that $(0, 0)$ is an element of it and any open ball around $(0, 0)$ contains some $(x^\cdot, y^\cdot)$ with $y^\cdot < 0$. Clearly though, $(x^\cdot, y^\cdot)$ is not an element of our set.

(e) Open: since the absolute value and identity functions are continuous (in fact, see the discussion below to see that any norm is continuous in the metric space it induces and how that relates to the "reverse triangle inequality"), we can use the argument outlined in the beginning of the previous part to show that $g(x, y) = |y| - x$ is continuous. Then:

$$\begin{aligned} & \{(x, y) \in \mathbb{R}^2 : |y| > x\} \\ = & \{(x, y) \in \mathbb{R}^2 : g(x, y) > 0\} \\ = & g^{-1}((0, \infty)) \end{aligned}$$

$g$ is continuous and $(0, \infty)$ is open, so our set is open as well.

The set is not closed since the sequence $\{0, 1/n\}$ is entirely contained within the set but its limit $(0, 0)$ is not an element of the set.

**Aside:** To show the continuity of the norm function in the metric space that it induces, let us first derive the "reverse triangle inequality" from the triangle inequality. Take some $x, y \in V$ with the norm $\|\cdot\|$ defined on it. Then:

$$\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$$

so

$$\|x\| - \|y\| \leq \|x - y\|.$$

Switching the roles of $x$ and $y$ in the above argument yields

$$\|y\| - \|x\| \leq \|x - y\|$$

so

$$\|x - y\| \geq |\, \|x\| - \|y\| \,|,$$

which is the *reverse triangle inequality*.

From here the continuity of $\|\cdot\|$ viewed as a function from $V$ to $\mathbb{R}$ follows, as for any $\varepsilon > 0$, set $\delta = \varepsilon$ and note that if $\|x - y\| < \delta$ then

$$|\, \|x\| - \|y\| \,| \leq \|x - y\| < \delta = \varepsilon.$$

2. Give examples of the following:

(a) A continuous function $f : S \to \mathbb{R}$, where $S$ is a closed subset of $\mathbb{R}$, that attains neither a maximum nor a minimum on $S$;

(b) A continuous function $f : S \to \mathbb{R}$, where $S$ is a closed and unbounded subset of $\mathbb{R}$, that attains both a maximum and a minimum on $S$;

(c) A continuous function $f : S \to \mathbb{R}$, where $S$ is a bounded subset of $\mathbb{R}$, that attains neither a maximum nor a minimum on $S$;

(d) A continuous function $f : S \to \mathbb{R}$, where $S$ is a bounded but not closed subset of $\mathbb{R}$, that attains both a maximum and a minimum on $S$;

(e) A discontinuous function $f : S \to \mathbb{R}$, where $S$ is a closed and bounded subset of $\mathbb{R}$, that attains neither a maximum nor a minimum on $S$;

(f) A discontinuous function $f : S \to \mathbb{R}$, where $S$ is a closed and bounded subset of $\mathbb{R}$, that attains both a maximum and a minimum on $S$.

**Solution:**

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$;

(b) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2$;

(c) $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = x$;

(d) $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = 2$;

(e) $f : [-1, 1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1/x \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$ ;

(f) $f : [-1, 1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1 \text{ if } x \geq 0 \\ 0 \text{ if } x < 0 \end{cases}$ .

3. Suppose $\{x_n\}$ is a Cauchy sequence in a metric space $X$, and some subsequence $\{x_{n_t}\}$ converges to $x \in X$. Prove that $\{x_n\}$ converges to $x$.

**Solution:** Let $d$ be the metric associated with the metric space. Fix $\varepsilon > 0$. The sequence $\{x_n\}$ is Cauchy, therefore:

$$\exists N(\varepsilon/2) : m, p > N(\varepsilon/2) \Rightarrow d(x_m, x_p) < \varepsilon/2 \tag{1}$$

The subsequence $\{x_{n_t}\}$ converges to $x$, therefore:

$$\exists N_T(\varepsilon/2) : p_t > N_T(\varepsilon/2) \Rightarrow d(x_{p_t}, x) < \varepsilon/2 \tag{2}$$

By (1) and (2), using the triangle inequality it follows that for any $x_{p_t} \in \{x_{n_t}\}$ such that $p_t > \max\{N(\varepsilon/2), N_T(\varepsilon/2)\}$:

$$m > \max\{N(\varepsilon/2), N_T(\varepsilon/2)\} \Rightarrow d(x_m, x) \leq d(x_m, x_{p_t}) + d(x_{p_t}, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Therefore $\{x_n\}$ converges to $x$.

4. Let $E, F \subseteq \mathbb{R}^n$. For $x \in E$, let $g(x) = \inf\{|x-y| : y \in F\}$. Prove that $g : E \to \mathbb{R}$ is continuous.

**Solution:** Let $x \in E$ be fixed and let $z$ be an arbitrary element of $E$. By the triangle inequality we get:

$$\begin{aligned} g(z) &= \inf\{|z-y| : y \in F\} \\ &\leq \inf\{(\underbrace{|z-x|}_{\text{fixed}} + |x-y|) : y \in F\} \\ &= |z-x| + \inf\{|x-y| : y \in F\} \\ &= |z-x| + g(x) \end{aligned}$$

Analogously, $g(x) \leq |x-z| + g(z)$. Therefore:

$$|g(z) - g(x)| \leq |z-x|$$

since $|x-z| = |z-x|$.

We need to show that for all positive $\varepsilon$ there exists a positive $\delta$ such that:

$$\forall z \in E : |x-z| < \delta \Rightarrow |g(x) - g(z)| < \varepsilon$$

Let $\varepsilon$ be given. Set $\delta = \varepsilon$. Consider any $z \in E$ such that $|x-z| < \delta = \varepsilon$. By the above: $|g(z) - g(x)| \leq |z-x| < \varepsilon$.

5. For all subsets $A, B$ of a metric space $(X, d)$ prove:

   (a) $A$ is both open and closed if and only if $\partial A = \varnothing$;

   (b) $\partial A = \partial(X \backslash A)$;

   (c) $\partial\partial A \subseteq \partial A$ (and give an example of a set A, such that this is a strict inclusion);

   (d) $\partial\partial\partial A = \partial\partial A$;

   (e) $\partial(A \cup B) \subseteq \partial A \cup \partial B$.

**Solution:**

   (a) This is immediate using result (3) and theorem 4.8 both on p. 60 in de la Fuente;

   (b) Applying the identity int $A \cup$ ext $A \cup \partial A = X$ (see result (2) on p. 60 in de la Fuente) to both $A$ and $X \backslash A$ and noting that int $A =$ ext $(X \backslash A)$, we see that $\partial A = \partial(X \backslash A)$;

   (c) Using part (b) and the fact that cl $A =$ int $A \cup \partial A$, we see that $\partial A =$ cl $A \cap$ cl $(X \backslash A)$. This suggests that the boundary of any set is closed as it is the intersection of two closed sets. Applying cl $A =$ int $A \cup \partial A$ again on $\partial A =$ cl $\partial A$ gives the desired result.

   An example of a set $A$, for which the inclusion is strict is the set of all rational numbers on $[0, 1] \in \mathbb{R}$. Then $\partial A = [0, 1]$ and $\partial\partial A = \{0, 1\}$

(d) Since the boundary of any set is closed, we'll prove the more general result: $\partial\partial S = \partial S$ for $S$ closed (in our problem $\partial A = S$).

One inclusion follows from the previous part. We need to show $\partial S \subseteq \partial\partial S$. Assume toward contradiction that we can find $x$ such that $x \in \partial S$ and $x \notin \partial\partial S$. By $x \in \partial S$, we have $\forall \varepsilon > 0 : B_\varepsilon(x)$ intersect both $S$ and $X\backslash S$. By $x \notin \partial\partial S$, $\exists \varepsilon_0 : B_{\varepsilon_0}(x)$ is entirely contained in $\partial S$ since $x \in \partial S$ by the definition of boundary. Therefore:

$$B_{\varepsilon_0}(x) \subseteq \partial S \subseteq S,$$

which is a contradiction.

(e) Let $x \in \partial(A \cup B)$. Then for all $\varepsilon > 0$, $B_\varepsilon(x)$ has a nonempty intersection with both $A \cup B$ and $X\backslash(A \cup B)$. If $B_\varepsilon(x)$ has a nonempty intersection with $A$ for all $\varepsilon > 0$ then, since $X\backslash(A \cup B) \subseteq X\backslash A$, $x \in \partial A$. Similarly, if $B_\varepsilon(x)$ has a nonempty intersection with $B$ for all $\varepsilon > 0$, $x \in \partial B$. In other words: $x \in \partial A \cup \partial B$.