Economics 204
Fall 2010
Problem Set 3 Solutions

1. In each case, give an example of a function f, continuous on S and such that $f(S) = T$, or else explain why there can be no so such f

   (a) $S = (0, 1), \quad T = (0, 1]$
   (b) $S = (0, 1), \quad T = (0, 1) \cup (1, 2)$
   (c) $S = \mathbf{R}, \quad T = \mathbf{Q}$
   (d) $S = [0, 1] \cup [2, 3], \quad T = \{0, 1\}$
   (e) $S = [0, 1] \times [0, 1], \quad T = \mathbf{R}^2$
   (f) $S = [0, 1] \times [0, 1], \quad T = (0, 1) \times (0, 1)$
   (g) $S = (0, 1) \times (0, 1), \quad T = \mathbf{R}^2$

   Solution.

   (a) $S = (0, 1), \quad T = (0, 1]$

   $$f(x) = \begin{cases} 2x & \text{if } x \in (0, \frac{1}{2}] \\ 1 & \text{if } x \in (\frac{1}{2}, 1). \end{cases}$$

   (b) $S = (0, 1), \quad T = (0, 1) \cup (1, 2)$
   No, a continuous functions maps a connected set to a connected set. However, in this case, S is connected and T is not.

   (c) $S = \mathbf{R}, \quad T = \mathbf{Q}$
   No, again a continuous functions maps a connected set to a connected set. However, in this case, S is connected and T is not.

   (d) $S = [0, 1] \cup [2, 3], \quad T = \{0, 1\}$

   $$f(x) = \begin{cases} 0 & \text{if } x \in [0, 1] \\ 1 & \text{if } x \in [2, 3]. \end{cases}$$

   (e) $S = [0, 1] \times [0, 1], \quad T = \mathbf{R}^2$
   No, a continuous functions sends a compact set to a compact set. However, in this case, S is compact and T is not.

   (f) $S = [0, 1] \times [0, 1], \quad T = (0, 1) \times (0, 1)$
   No, a continuous functions sends a compact set to a compact set. However, in this case, S is compact and T is not.

   (g) $S = (0, 1) \times (0, 1), \quad T = \mathbf{R}^2$

   $$f(x, y) = (\cot \pi x, \cot \pi y).$$

2. Let $X = C([0,1])$, $d(f,g) = \max_t |f(t) - g(t)|$. Show that (X, d) is not compact.

   Solution. Recall the sequential definition of compactness: every sequence have a convergent subsequence. So, we will show that (X, d) is not compact by giving an example of one such sequence. Consider the following sequence of constant functions $\{f_n = n \ \forall n\}$, which clearly contains no convergent subsequence. Therefore, $C([0,1])$ with a supremum metric is not compact.

3. Show that any sequence $\{x_n\}$ in a compact metric space X, that has a unique cluster point x, converges to x.

   Solution. Lets prove this statement by contradiction. So, lets assume that that $\{x_n\} \nrightarrow x$. Then, we can find an $\epsilon > 0$ such that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, the elements of which belong to the set $X \setminus B_\epsilon(x)$. Since X is a compact metric space, $\{x_{n_k}\}$ has a cluster point $\tilde{x} \neq x$. But $X \setminus B_\epsilon(x)$ is closed, thus, $\tilde{x} \in X \setminus B_\epsilon(x)$, so $\tilde{x} \neq x$. We obtained two different cluster points x and $\tilde{x}$ which is a contradiction.

4. Assume $f : S \to T$ is uniformly continuous on S, where S and T are metric spaces. If $\{x_n\}$ is any Cauchy sequence in S, prove that $\{f(x_n)\}$ is a Cauchy sequence in T. Provide an example to show that the statement is not true if f is just continuous.

   Solution. Lets assume that d is a metric on T and ρ is a metric on S. We need to show that given an $\epsilon > 0$ there is a $N(\epsilon) \in \mathbf{N}$ such that for any $n, m \geq N(\epsilon)$ we have $d(f(x_n), f(x_m)) < \epsilon$. So, fix some $\epsilon > 0$. Since f is uniformly continuous there is $\delta > 0$ such that whenever $\rho(x,y) < \delta$, $x, y \in S$ we have $d(f(x), f(y)) < \epsilon$. By our assumptions, $\{x_n\}$ is a Cauchy sequence, therefore, for that $\delta(\epsilon) > 0$ there is a positive integer $N(\delta) \in \mathbf{N}$ such that for any $n, m \geq N(\delta)$ we have $\rho(x_n, x_m) < \delta$ which implies $d(f(x_n), f(x_m)) < \epsilon$. Thus, $\{f(x_n)\}$ is Cauchy.

   Now, let $x_n = \frac{1}{n}$, $f(x) = \frac{1}{x}$, $S = (0,1]$ and $T = \mathbf{R}$ with usual Euclidean metric on both spaces. It is easy to see that while $\{x_n\}$ is Cauchy, $\{f(x_n) = n \ \forall n\}$ is not because $\frac{1}{x}$ is not uniformly continuous.

5. Give an example of each of the following:

   (a) a complete metric space that is bounded but not compact.

   Solution. Recall the discrete metric defined (on $\mathbf{R}$) as follows:

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Then, it is easy to see that $(\mathbf{R}, d)$ is a bounded (and complete) metric space, but it is not compact. Since every point is open with discrete metric, it follows that the cover $\{x_\alpha\}_{\alpha \in \mathbf{R}}$ whose elements are singletons, has no finite sub-cover.

To see completeness, note that there are no Cauchy sequences in $(\mathbf{R}, d)$ with infinitely many distinct terms. Else, for every $x_m \neq x_n \Rightarrow d(x, y) = 1$. Hence, the only Cauchy sequences are those that are constant after some (finite) index n. As constant sequences vacuously converge to their constant value, it follows that the given metric space is complete.

(b) a metric space with the property that none of its closed balls is complete.

Solution. Consider $\mathbf{Q} \subset \mathbf{R}$ with open sets "inherited" from $\mathbf{R}$ under the usual absolute value metric. In other words, a subset V of $\mathbf{Q}$ is open if and only if there exists an open $U \subset \mathbf{R}$ such that $V = U \cap \mathbf{R}$. It is easy to see that none of the closed balls could possibly be complete because we can't capture irrational limit points.

6. Some practice with connectedness

(a) A space is *totally disconnected* if its only connected subsets are one-point sets. Show that if X is endowed with *discrete metric*, then X is totally disconnected. Does the converse hold?

Solution. Notice that with discrete metrics every set is open. So, if $A \subset X$ contains more then one element then for every $x \in A$, then two open sets $\{x\}$ and $A \setminus \{x\}$ form a separation of A. Thus, the only connected sets are singletons.

The converse is not necessarily true. Consider $\mathbf{Q} \subset \mathbf{R}$ with open sets "inherited" from $\mathbf{R}$ under the usual absolute value metric. In other words, a subset V of $\mathbf{Q}$ is open if and only if there exists an open $U \subset \mathbf{R}$ such that $V = U \cap \mathbf{R}$. Then $\mathbf{Q} \subset \mathbf{R}$ is totally disconnected (to see this pick an irrational point t and consider separation $\mathbf{Q} = (-\infty, t) \cup (t, +\infty)$).

(b) Show that a topological space X is connected if and only if every continuous function $f : X \to \{0, 1\}$ is constant.[1]

Solution.

($\Rightarrow$) Assume that X is connected and let $f : X \to \{0, 1\}$ be any continuous function. We claim f is constant. Proceeding by contradiction, assume not. Then, by continuity, $U = f^{-1}(1), V = f^{-1}(0)$ are open subsets of X. Moreover, they are non-empty if f is non-constant and disjoint. Since $U \cup V = X$ we obtain a contradiction to the connectedness of X.

[1] $\{0, 1\}$ is endowed with the *discrete metric.*

($\Leftarrow$) We prove the contrapositive: If X is not connected, then there is a non-constant continuous function $f : X \to \{0, 1\}$. If X is not connected, then we may write $X = U \cup V$, where U, V are open subsets of X that form a separation. Define $f : X \to \{0, 1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \in V \end{cases}$$

Clearly, this gives a well-defined, continuous function from X to $\{0, 1\}$, proving the contrapositive.

(c) Let X be a connected subset of a metric space S. Let Y be a subset of S such that $X \subseteq Y \subseteq \bar{X}$, where $\bar{X}$ is the closure of X. Prove that Y is also connected using the result from part (b) of this exercise. Provide counter-example showing that converse is not true.

Solution. Lets use the result we have just proven. So consider a two-valued function f on Y. It is easy to see that the restriction of f to Y, $f|_Y$ is also two valued.[2] Given our assumptions, f is constant on X and without any loss of generality lets assume $f = 0$ on X. Now, lets consider $y \in Y \setminus X$. Since $X \subseteq Y \subseteq \bar{X}$, y must be a limit point of X. This means that there is a sequence $\{x_n\} \subset X$ such that $x_n \to y$. Clearly, $f(x_n) = 0$ for all n. f is a continuous function, so

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(y) = 0$$

Since point y was arbitrary $f = 0$ on Y and by previous result we conclude that Y is connected.

To see that converse does not hold consider $\mathbf{Q}$, a disconnected set with connected closure.

7. Let $X \subseteq \mathbf{E}^n$, $Y \subseteq \mathbf{E}^m$. Suppose $\Psi : X \to 2^Y$ is a correspondence. Define $\Psi^+(V)$ to be *upper (or strong)* inverse of $V \subseteq Y$ if

$$\Psi^+(V) = \{x \in X : \quad \Psi(x) \subseteq V\}$$

and $\Psi^-(V)$ to be *lower (or weak)* inverse of $V \subseteq Y$ if

$$\Psi^-(V) = \{x \in X : \quad \Psi(x) \cap V \neq \emptyset\}.$$

Using these definitions show that

[2] The restriction $f|_Y$ is the function from Y to $\{0, 1\}$ such that $f|_Y(x) = f(x)$ for all $x \in Y$.

(a) For every $V \subseteq Y$, $\Psi^+(V) = [\Psi^-(V^c)]^c$

Solution. Note that for all $x \in X$ either $\Psi(x) \subseteq V$ or $\Psi(x) \cap V^c \neq 0$ but not both. Therefore,

$$\Psi^+(V) \cup \Psi^-(V^c) = X$$
$$\Psi^+(V) \cap \Psi^-(V^c) = \emptyset$$

Combining these two equalities we obtain the result we seek.

(b) $\Psi(x)$ is uhc $\iff$ $\Psi^-(V)$ is closed for every closed set V

Solution.
($\Longrightarrow$) Assume that $\Psi(x)$ is uhc and let V be a closed set in Y. We need to show that $\Psi^-(V)$ is closed in X. V^c is open and because the strong inverse of any open set is open, $\Psi^+(V^c)$ is open. By the result in the previous part

$$\Psi^-(V) = [\Psi^+(V^c)]^c$$

thus, $\Psi^-(V)$ is closed as a complement of an open set.

($\Longleftarrow$) Assume $\Psi^-(V)$ is closed for every closed set V. We need to show that $\Psi(x)$ is uhc. We do this by showing that $\Psi^+(W)$ is open for any open W in Y. So, take W open in Y, then W^c is closed and $\Psi^-(W^c)$ is closed in X. By the result in the previous part

$$\Psi^+(W) = [\Psi^-(W^c)]^c$$

thus, $\Psi^+(V)$ is open as a complement to a closed set.

(c) $\Psi(x)$ is lhc $\iff$ $\Psi^+(V)$ is closed for every closed set V

Solution.
($\Longrightarrow$) Assume that $\Psi(x)$ is lhc and let V be a closed set in Y. We need to show that $\Psi^+(V)$ is closed in X. V^c is open and because the weak inverse of any open set is open, $\Psi^-(V^c)$ is open. By the result in the previous part

$$\Psi^+(V) = [\Psi^-(V^c)]^c$$

thus, $\Psi^+(V)$ is closed as a complement of the open set.

($\Longleftarrow$) Assume $\Psi^+(V)$ is closed for every closed set V. We need to show that $\Psi(x)$ is lhc. We do this by showing that $\Psi^-(W)$ is open for any open W in Y. So, take W open in Y, then W^c is closed and $\Psi^+(W^c)$ is closed in X. By the result in the previous part

$$\Psi^-(W) = [\Psi^+(W^c)]^c$$

thus, $\Psi^-(V)$ is open as a complement to a closed set.

8. Let $X \subseteq \mathbf{E}^n$, $Y \subseteq \mathbf{E}^m$. Suppose $\Psi : X \to 2^Y$ is uhc and compact valued-correspondence. Show that $\Psi(K)$ is compact if K is compact.

   Solution. Let $\{y_n\}$ be a sequence in $\Psi(K)$. We have to show that $\{y_n\}$ has a convergent subsequence with a limit in $\Psi(K)$. For every $\{y_n\}$ there is an x_n with $y_n \in \Psi(x_n)$. Since K is compact, the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\} \to x \in K$. Now we appeal to the sequential characterization of uhc (Theorem 12 in lecture notes 7) to claim that subsequence $\{y_{n_k}\}$ has a further subsequence $\{y_{n_{k_l}}\}$ that converges to $y \in \Psi(x) \subseteq \Psi(K)$. Thus, we have shown that the original sequence $\{y_n\}$ has a convergent subsequence.