Economics 204
Fall 2010
Problem Set 3
Due Friday, August 6 in Lecture

1. In each case, give an example of a function f, continuous on S and such that $f(S) = T$, or else explain why there can be no so such f
   (a) $S = (0, 1), \quad T = (0, 1]$
   (b) $S = (0, 1), \quad T = (0, 1) \cup (1, 2)$
   (c) $S = \mathbf{R}, \quad T = \mathbf{Q}$
   (d) $S = [0, 1] \cup [2, 3], \quad T = \{0, 1\}$
   (e) $S = [0, 1] \times [0, 1], \quad T = \mathbf{R}^2$
   (f) $S = [0, 1] \times [0, 1], \quad T = (0, 1) \times (0, 1)$
   (g) $S = (0, 1) \times (0, 1), \quad T = \mathbf{R}^2$
2. Let $X = C([0, 1])$, $d(f, g) = \max_t |f(t) - g(t)|$. Show that (X, d) is not compact.
3. Show that any sequence $\{x_n\}$ in a compact metric space X, that has a unique cluster point x, converges to x.
4. Assume $f : S \to T$ is uniformly continuous on S, where S and T are metric spaces. If $\{x_n\}$ is any Cauchy sequence in S, prove that $\{f(x_n)\}$ is a Cauchy sequence in T. Provide an example to show that the statement is not true if f is just continuous.
5. Give an example of each of the following:
   (a) a complete metric space that is bounded but not compact.
   (b) a metric space with the property that none of its closed balls is complete.
6. Some practice with connectedness
   (a) A space is *totally disconnected* if its only connected subsets are one-point sets. Show that if X is endowed with *discrete metric*, then X is totally disconnected. Does the converse hold?
   (b) Show that a topological space X is connected if and only if every continuous function $f : X \to \{0, 1\}$ is constant.[1]
   (c) Let X be a connected subset of a metric space S. Let Y be a subset of S such that $X \subseteq Y \subseteq \bar{X}$, where $\bar{X}$ is the closure of X. Prove that Y is also connected using the result from part (b) of this exercise. Provide counter-example showing that converse is not true.

[1] $\{0, 1\}$ is endowed with the *discrete metric*.

7. Let $X \subseteq \mathbf{E}^n$, $Y \subseteq \mathbf{E}^m$. Suppose $\Psi : X \to 2^Y$ is a correspondence. Define $\Psi^+(V)$ to be *upper (or strong)* inverse of $V \subseteq Y$ if

$$\Psi^+(V) = \{x \in X : \quad \Psi(x) \subseteq V\}$$

and $\Psi^-(V)$ to be *lower (or weak)* inverse of $V \subseteq Y$ if

$$\Psi^-(V) = \{x \in X : \quad \Psi(x) \cap V \neq \emptyset\}.$$

Using these definitions show that

(a) For every $V \subseteq Y$, $\Psi^+(V) = [\Psi^-(V^c)]^c$

(b) $\Psi(x)$ is uhc $\iff$ $\Psi^-(V)$ is closed for every closed set V

(c) $\Psi(x)$ is lhc $\iff$ $\Psi^+(V)$ is closed for every closed set V

8. Let $X \subseteq \mathbf{E}^n$, $Y \subseteq \mathbf{E}^m$. Suppose $\Psi : X \to 2^Y$ is uhc and compact valued-correspondence. Show that $\Psi(K)$ is compact if K is compact.