Economics 204
Fall 2010
Problem Set 4 Suggested Solutions

1. Prove that any set of pairwise orthogonal nonzero vectors is linearly independent.

   Solution: Let $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ be a set of pairwise orthogonal nonzero vectors. Assume
$$\mathbf{0} = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \cdots + \alpha_n \mathbf{x}_n$$
   This implies:
$$0 = \mathbf{0}^T \mathbf{x}_j = \sum_{i=1}^{n} \alpha_i \mathbf{x}_i^T \mathbf{x}_j = \alpha_j \mathbf{x}_j^T \mathbf{x}_j$$
   for every $j = 1, 2, \dots, n$. Since $\mathbf{x}_j \neq \mathbf{0}$, this implies $\alpha_j = 0$ for every $j = 1, 2, \dots, n$. We conclude that the set of vectors is linearly independent.

2. Prove that if U, V and $U \cup V$ are subspaces of a vector space W, then either $U \subseteq V$ or $V \subseteq U$.

   Solution: Suppose toward contradiction that $U \not\subseteq V$ and $V \not\subseteq U$. Then there exist vectors u and v such that:
$$u \in U \text{ and } u \notin V, \quad v \in V \text{ and } v \notin U$$
   Consider $u + v$. As we are assuming $U \cup V$ is a subspace, $U \cup V$ is closed under addition. Hence $u + v \in U \cup V$ and so $u + v \in U$ or $u + v \in V$. However, if $u + v \in U$, then $v = (u + v) - u \in U$, which is a contradiction since U is a subspace. Similarly, we reach a contradiction if we assume $u + v \in V$.

3. $T : M_{2\times 2} \to M_{2\times 3}$ is defined by:
$$T \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} + a_{21} & a_{11} + 3a_{22} & 0 \\ a_{11} - a_{12} & a_{12} + a_{21} & 0 \end{pmatrix}$$
   Determine $\ker T$, $\dim(\ker T)$, and $\operatorname{rank} T$. Is T one-to-one, onto, both or neither?

   Solution: $\ker T$ is the set of all 2×2 matrices of the form
$$\begin{pmatrix} x & x \\ -x & -x/3 \end{pmatrix}$$
   which is a one-dimensional space. Rank $T = 3$ because $\dim(\ker T) + \operatorname{rank} T = \dim M_{2\times 2} = 4$. T is not one-to-one by Theorem 2.13 on p. 126 in de la Fuente. T is not onto either since the rank of T is 3 and not 6. Alternatively, you can directly verify that T fails to be both one-to-one and onto.

4. (a) Suppose that V is a finite-dimensional vector space. Show that a linear transformation $T \in L(V)$ (i.e. $T : V \to V$) is invertible if and only if $\ker T = \{0\}$.
   (b) Suppose again that V is finite-dimensional and $T, S \in L(V)$. Using part (a), prove that $T \circ S$ is invertible if and only if both T and S are invertible.

Solution:

(a) From applying Theorem 2.9 on p.124 in de la Fuente (the *Rank-Nullity Theorem*) to the linear operator T (linear operator is a linear functions from a space into itself), we see that having a kernel of dimension zero is equivalent to the following equality:

$$\text{rank } T = \dim \text{im } T = \dim V.$$

In other words, a linear operator has a kernel of dimension zero if and only if it is surjective. Using then Theorem 2.13 on p.126 gives us that a linear operator has a kernel of dimension zero if and only if it is both surjective and injective or, equivalently, invertible.

Aside: Notice that this result is not true for infinite-dimensional vector spaces. Consider the following counterexample: $T : \ell^2 \to \ell^2$ (recall we defined ℓ^2 to be the space of all square-summable sequences) defined by:

$$T((x_1, x_2, x_3 \ldots)) = (0, x_1, x_2, x_3, \ldots)$$

It is easy to verify that T is linear, and that $\ker T = \{0\}$. But clearly T is not onto: $\text{im}\, T = \{y \in \ell^2 : y_1 = 0\}$, which is a proper vector subspace of ℓ^2. So T is not invertible. How is this possible? Notice we always have (regardless of dimension of X) that if $T \in L(X, X)$, then

$$X/\ker T \cong \text{im}\, T$$

When T is 1-1, so $\ker T = \{0\}$, then $X/\ker T = X/\{0\} = X$. So we do know in general that the image of T is isomorphic to X, with T as the isomorphism. The problem is that an infinite-dimensional vector space can be isomorphic to a proper vector subspace of itself. Above shows you an example of this.

(b) Assume that $T \circ S$ is invertible. We will first check that S is invertible. By part (a), it suffices to check that $\ker S = \{0\}$. If $\exists w \in V, w \neq 0$, then: $TSw = T0 = 0$. Hence, $T \circ S$ has a non-zero kernel, a contradiction by (a). Therefore S is invertible.

If T is not invertible, assume toward contradiction that there is some $v \in V$ with $Tv = 0$ ($v \neq 0$). Since S is invertible, it is surjective. Thus, we can find a $w \in V$ such that $Sw = v$, which implies that $(TS)w = T(Sw) = Tv = 0$. This, again, contradicts the invertibility of TS.

We now prove the converse: assuming that T, S are both invertible, we wish to check that $T \circ S$ is invertible. We again check that $\ker(T \circ S) = \{0\}$. To see that this is the case, we note that

$$w \in \ker(T \circ S) \Rightarrow Sw \in \ker T \Rightarrow Sw = 0 \Rightarrow w \in \ker S \Rightarrow w = 0.$$

5. (a) Prove that the eigenvalues of any upper or lower triangular matrix A are the diagonal entries of A;
   (b) Show that the eigenspace of any matrix A belonging to an eigenvalue λ_i (see de la Fuente, p. 147 for a definition) is a vector space;
   (c) Show that if λ is an eigenvalue of A then λ^k is an eigenvalue of A^k for $k \in \mathbb{N}$;
   (d) Show that if λ is an eigenvalue of the invertible matrix A then $1/\lambda$ is an eigenvalue of A^{-1}.

Solution:

(a) Let us denote the diagonal elements of A by $\{t_{11}, t_{22}, t_{33}, \ldots, t_{nn}\}$. Using induction on the size of the matrix, it is easy to show by directly computing the determinant through Laplace expansion that the determinant of any triangular (or diagonal) matrix is the product of its diagonal elements. Thus the characteristic polynomial for A is:

$$\det(A - \lambda I) = (t_{11} - \lambda)(t_{22} - \lambda) \cdots (t_{nn} - \lambda),$$

so the eigenvalues are the t_{ii}'s.

(b) To show that the eigenspace is a vector space, we only need to check the existence of additive identity and inverse elements. Since the eigenspace contains the $\mathbf{0}$ vector by definition, we only need to verify the existence of inverse elements. Denote the eigenspace by E_i. Let $v \in E_i$. Then $Av = \lambda_i v$. Multiplying both sides by -1 gives us $A(-v) = \lambda_i(-v) \Rightarrow (-v) \in E_i$.

(c) We use induction to show not only that λ^k is an eigenvalue of A^k, but also that any eigenvector v corresponding to the eigenvalue λ for A also corresponds to λ^k for A^k. The base step ($k = 1$) is trivial. For the induction step, assume $Av = \lambda v$ and $A^k v = \lambda^k v$. Now consider $A^{k+1}v$:

$$A^{k+1}v = A^k(Av) = A^k(\lambda v) = \lambda(A^k v) = \lambda(\lambda^k v) = \lambda^{k+1}v$$

(d) Let $Tv = \lambda v$. Premultiply both sides by T^{-1}:

$$T^{-1}Tv = T^{-1}\lambda v \Rightarrow v = \lambda T^{-1}v \Rightarrow T^{-1}v = (1/\lambda)v$$

6. Let Θ be the set of all continuous functions whose domain is the unit interval $[0, 1]$ and range is the real line $\mathbb{R}$:

$$\Theta \equiv \{ f(x) \,|\, f : [0,1] \to \mathbb{R} \,\text{and}\, f \in C^0 \}$$

Let Φ be the subset consisting of all real polynomials (whose domain is restricted to the unit interval) of degree at most two:

$$\Phi \equiv \{ a + bx + cx^2 \,|\, a, b, c \in \Re \}$$

Note that the set Θ is a vector space over the field of real numbers and the subset Φ is a proper subspace.

(a) Are the vectors $\{ x, (x^2 - 1), (x^2 + 2x + 1) \}$ linearly independent over $\mathbb{R}$?

(b) Find a Hamel basis for the subspace Φ.

(c) What is the dimension of Φ ? What is the dimension of Θ ?

Solution:

(a) Apply the usual test for independence of vectors. Solve for a, b, and c such that

$$ax + b(x^2 - 1) + c(x^2 + 2x + 1) = 0 \Leftrightarrow (c - b) + (a + 2c)x + (b + c)x^2 = 0$$

We obtain the following system of equations:

$$\begin{aligned} b + c &= 0 \\ a + 2c &= 0 \\ c - b &= 0 \end{aligned}$$

Solving it, we get $a = b = c = 0$. Thus, the three vectors are linearly independent over $\mathbb{R}$.

(b) $\{x, x^2 - 1, x^2 + 2x + 1\}$ is one possible basis. $\{1, x, x^2\}$ is another one.

(c) 3 and ∞, respectively. The dimension of Φ follows immediately from parts (a) and (b). To see that the dimension of Θ is infinite, note that the set of vectors $\{1, x, x^2, x^3, \ldots\}$ is linearly independent over $\mathbb{R}$. Hence, Θ has a vector subspace of infinite dimension. It follows that Θ itself must have infinite dimension.