1. Prove that any set of pairwise orthogonal nonzero vectors is linearly independent.

2. Prove that if $U$, $V$ and $U \cup V$ are subspaces of a vector space $W$, then either $U \subseteq V$ or $V \subseteq U$.

3. $T : M_{2 \times 2} \rightarrow M_{2 \times 3}$ is defined by:
   \[
   T \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} + a_{21} & a_{11} + 3a_{22} & 0 \\ a_{11} - a_{12} & a_{12} + a_{21} & 0 \end{pmatrix}
   \]
   Determine $\ker T$, $\dim(\ker T)$, and $\text{rank} T$. Is $T$ one-to-one, onto, both or neither?

4. (a) Suppose that $V$ is a finite-dimensional vector space. Show that a linear transformation $T \in L(V)$ (i.e. $T : V \rightarrow V$) is invertible if and only if $\ker T = \{0\}$.
   (b) Suppose again that $V$ is finite-dimensional and $T, S \in L(V)$. Using part (a), prove that $T \circ S$ is invertible if and only if both $T$ and $S$ are invertible.

5. (a) Prove that the eigenvalues of any upper or lower triangular matrix $A$ are the diagonal entries of $A$;
   (b) Show that the eigenspace of any matrix $A$ belonging to an eigenvalue $\lambda_i$ (see de la Fuente, p. 147 for a definition) is a vector space;
   (c) Show that if $\lambda$ is an eigenvalue of $A$ then $\lambda^k$ is an eigenvalue of $A^k$ for $k \in \mathbb{N}$;
   (d) Show that if $\lambda$ is an eigenvalue of the invertible matrix $A$ then $1/\lambda$ is an eigenvalue of $A^{-1}$.

6. Let $\Theta$ be the set of all continuous functions whose domain is the unit interval $[0, 1]$ and range is the real line $\mathbb{R}$:
   \[
   \Theta \equiv \{ f(x) \mid f : [0, 1] \rightarrow \mathbb{R} \text{ and } f \in C^0 \}
   \]
   Let $\Phi$ be the subset consisting of all real polynomials (whose domain is restricted to the unit interval) of degree at most two:
   \[
   \Phi \equiv \{ a + bx + cx^2 \mid a, b, c \in \mathbb{R} \}
   \]
   Note that the set $\Theta$ is a vector space over the field of real numbers and the subset $\Phi$ is a proper subspace.
   (a) Are the vectors $\{ x, (x^2 - 1), (x^2 + 2x + 1) \}$ linearly independent over $\mathbb{R}$?
   (b) Find a Hamel basis for the subspace $\Phi$.
   (c) What is the dimension of $\Phi$? What is the dimension of $\Theta$?