Economics 204
Fall 2010
Problem Set 4
Due Tue, Aug 10 in Lecture

1. Prove that any set of pairwise orthogonal nonzero vectors is linearly independent.

2. Prove that if U, V and $U \cup V$ are subspaces of a vector space W, then either $U \subseteq V$ or $V \subseteq U$.

3. $T : M_{2\times 2} \to M_{2\times 3}$ is defined by:
$$T\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}+a_{21} & a_{11}+3a_{22} & 0 \\ a_{11}-a_{12} & a_{12}+a_{21} & 0 \end{pmatrix}$$
Determine $\ker T$, $\dim(\ker T)$, and $\operatorname{rank} T$. Is T one-to-one, onto, both or neither?

4. (a) Suppose that V is a finite-dimensional vector space. Show that a linear transformation $T \in L(V)$ (i.e. $T : V \to V$) is invertible if and only if $\ker T = \{0\}$.
   (b) Suppose again that V is finite-dimensional and $T, S \in L(V)$. Using part (a), prove that $T \circ S$ is invertible if and only if both T and S are invertible.

5. (a) Prove that the eigenvalues of any upper or lower triangular matrix A are the diagonal entries of A;
   (b) Show that the eigenspace of any matrix A belonging to an eigenvalue λ_i (see de la Fuente, p. 147 for a definition) is a vector space;
   (c) Show that if λ is an eigenvalue of A then λ^k is an eigenvalue of A^k for $k \in \mathbb{N}$;
   (d) Show that if λ is an eigenvalue of the invertible matrix A then $1/\lambda$ is an eigenvalue of A^{-1}.

6. Let Θ be the set of all continuous functions whose domain is the unit interval $[0,1]$ and range is the real line $\mathbb{R}$:
$$\Theta \equiv \{ f(x) \mid f : [0,1] \to \mathbb{R} \text{ and } f \in C^0 \}$$
Let Φ be the subset consisting of all real polynomials (whose domain is restricted to the unit interval) of degree at most two:
$$\Phi \equiv \{ a + bx + cx^2 \mid a, b, c \in \Re \}$$
Note that the set Θ is a vector space over the field of real numbers and the subset Φ is a proper subspace.
   (a) Are the vectors $\{ x, (x^2-1), (x^2+2x+1) \}$ linearly independent over $\mathbb{R}$?
   (b) Find a Hamel basis for the subspace Φ.
   (c) What is the dimension of Φ ? What is the dimension of Θ ?