Economics 204
Fall 2010
Problem Set 5 Solutions

1. Recall that a reflection across the x-axis can be achieved with the transformation $(x, y) \to (x, -y)$. Derive a transformation, T, which reflects a point across the line $y = 3x$.

   (a) First, calculate the action of T on the points $(1, 3)$ and $(-3, 1)$.

   Solution. Since $(1, 3)$ lies on the line $y = 3x$ it is unchanged by T so we know that $T(1, 3) = (1, 3)$. The slope of the line $y = 3x$ is 3 and the slope of the vector $(-3, 1)$ is $-\frac{1}{3}$. Because the vector is perpendicular with the line, reflecting it across the line takes is to $(3, -1)$, so $T(-3, 1) = (3, -1)$.

   (b) Next, write the matrix representation of T using these two vectors as bases.

   Solution. Let
$$V = \{v_1, v_2\} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\}.$$
   From (1a) we know that $T(v_1) = v_1 = 1 \cdot v_1 + 0 \cdot v_2$ and $T(v_2) = -v_2 = 0 \cdot v_1 - 1 \cdot v_2$. So we write
$$P = Mtx_V(T) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

   (c) Find S and S^{-1}, the matrices that changes coordinates under this basis to standard coordinates and back again.

   Solution. We can easily find S, the matrix that changes coordinates in V to coordinates under the standard basis, E, because we already expressed the basis vectors v_1 and v_2 in terms of standard basis coordinates. We have
$$S = Mtx_{E,V}(id) = [\ v_1 \quad v_2\ ] = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}.$$
   The matrix that changes coordinates from E to V is simply the inverse of S. So
$$S^{-1} = Mtx_{V,E}(id) = \frac{1}{10} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1/10 & 3/10 \\ -3/10 & 1/10 \end{bmatrix}.$$

(d) Write the matrix representation of T in the standard basis.

Solution. One way to find $Mtx_E(T)$ would be to calculate $T(e_1)$ and $T(e_2)$ put the coordinates of these vectors in the columns of $Mtx_E(T)$. However, since we are not given a formula for T—just a description of its action—it takes a little work to find $T(e_1)$ and $T(e_2)$. (The reason we use V as a basis is because it the action of T on these basis vectors is straightforward.) Instead, we will apply the commutative diagram by changing coordinates to V, applying T, and changing back to E. In other words,

$$Mtx_E(T) = Mtx_{E,V}(id) \cdot Mtx_V(T) \cdot Mtx_{V,E}(id) = SPS^{-1}$$

and this equals

$$\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/10 & 3/10 \\ -3/10 & 1/10 \end{bmatrix} = \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix}.$$

(e) Use point $(-3, 1)$ to verify the commutative diagram.

Solutions. We established in $(1a)$ that $T(-3, 1) = (3, -1)$ so now we will verify that multiplying $SPS^{-1}(-3, 1)^T$ yields the same result. Writing out this computation, we have

$$\begin{aligned} SPS^{-1}(-3,1)^T =&(SP)(S^{-1}(-3,1)^T) = (SP)(0,1)^T = \\ =&S(P(0,1)^T) = S(0,-1)^T, \end{aligned}$$

and this equals

$$\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = (3, -1)^T.$$

2. Similarity and eigenvalues

(a) Which matrices are similar to the identity matrix? to zero matrix?

Solution. The only matrix similar to the zero matrix is itself: $PZP^{-1} = PZ = Z$. The only matrix similar to the identity matrix is itself: $PIP^{-1} = PP^{-1} = I$.

(b) What would your answers to (2a) suggest about similarity of the matrix of the form cI for some scalar c? Or, about a similarity of the diagonal matrix?

Solution. Because scalar multiples can be brought out of a matrix

$$P(cI)P^{-1} = cPIP^{-1} = cI$$

the only matrix similar to cI is itself (notice that zero and identity matrices are just the special cases with $c = 0$ and $c = 1$). This is not true, however about the diagonal matrix as the following example shows.

$$\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -5 & -4 \\ 2 & 1 \end{bmatrix}$$

(c) Show that if $T - \lambda I$ and N are similar matrices then T and $N + \lambda I$ are also similar.

Solution. If $N = P(T - \lambda I)P^{-1}$ then $N = PTP^{-1} - P(\lambda I)P^{-1}$. The diagonal matrix λI commutes with anything, so

$$P(\lambda I)P^{-1} = PP^{-1}(\lambda I) = \lambda I.$$

Thus, we have $N = PTP^{-1} - \lambda I$ and consequently $N + \lambda I = PTP^{-1}$. So, not only they are similar, they are similar via the same P.

3. Let A and B denote symmetric real $n \times n$ matrices, such that $AB = BA$. Prove that A and B have a common eigenvector in $\mathbb{R}^n$.

Solution. Since both A and B are symmetric real matrices, their eigenvalues and eigenvectors should be real as well. Let $\vec{v} \in V$ be an eigenvector of A corresponding to the eigenvalue λ with V being its eigenspace. We have $A\vec{v} = \lambda\vec{v} \implies B(A\vec{v}) = B(\lambda\vec{v})$. Since A and B commute, we have $A(B\vec{v}) = \lambda B\vec{v}$. Since $\vec{v}$ was arbitrary, it shows that V is invariant under B, i.e. B operating on the vectors in V leaves them in V. Now consider a linear transformation $T : V \to V$ which is a restriction of B to V, $T = B|_V$, in other words $T\vec{v} = B\vec{v}$ for all $\vec{v} \in V$. Consider $\vec{v^*}$, which is an eigenvector of T associated with eigenvalue μ. Note that $B\vec{v^*} \in V$ and $B\vec{v^*} = \mu\vec{v^*} \implies \mu\vec{v^*} \in V$ giving us the result we seek.

4. Consider the following quadratic forms:

$$f(x, y) = 5x^2 + 2xy + 5y^2,$$
$$g(x, y) = 10xy$$

Answer the following questions for each of these forms:

(a) Find a symmetric matrix M such that the form equals $[x\ y]\ M \begin{bmatrix} x \\ y \end{bmatrix}$.

Solution. We seek a, b, c, and d such that:

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 5x^2 + 2xy + 5y^2$$

The diagonal elements of the matrix are the coefficients of the squared terms and the off diagonal elements are half of the crossed terms. This gives us

$$M_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}.$$

For the second quadratic form, $g(x, y) = 10xy$, we find:

$$M_g = \begin{bmatrix} 0 & 5 \\ 5 & 0 \end{bmatrix}.$$

(b) Find the eigenvalues of the form.

Solution. We compute the characteristic polynomial for each matrix, set it equal to zero, and solve. We find that the eigenvalues of M_f are $6, 4$ and $+5, -5$ for the M_g.

(c) Find an orthonormal basis of eigenvectors.

Solution. We find that $(1, 1)$ and $(-1, 1)$ form a basis of eigenvectors for M_f. To normalize, we divide both vectors by their respective lengths, which yields an orthonormal basis of eigenvectors:

$$\{v_1, v_2\} = \left\{ \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \right\}.$$

Repeating the same process for matrix M_g yields

$$\{w_1, w_2\} = \left\{ \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \right\}.$$

(d) Find a unitary matrix S such that $M = S^{-1}DS$, where D is a diagonal matrix.

Solution. We do this first for matrix M_f. Note that $S^{-1} = (Mtx)_{U,V}(id)$, where U is the standard basis, so the columns are just the eigenvectors:

$$S^{-1} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

Since the columns of S^{-1} are orthonormal, it follows that S^{-1} is a unitary matrix. Since S^{-1} is unitary, $S = (S^{-1})^{-1} = (S^{-1})^T$, so

$$S = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

Since the eigenvectors of matrix M_g are the same, so will be matrices S and S^{-1}.

(e) Describe the level sets of the form and state whether the form has a local maximum, local minimum, or neither at $(0,0)$. (Level sets are solutions to $f(x,y) = c$ for some $c \in \mathbb{R}$.)

Solution. The quadratic form f is associated with matrix M_f which has two positive eigenvalues. This means that the level sets are ellipses and that there is a local minimum at the origin.

The maximum value of the form on the unit circle is simply the norm of M, which is equal to the largest of the absolute values of the eigenvalues, which is 6. Similarly, the minimum value of the form on the unit circle is 4. To obtain level sets of the form, we convert to the basis of orthonormal eigenvectors v_1 and v_2 and write

$$f(\gamma_1, \gamma_2) = (\gamma_1, \, \gamma_2) \; D \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$$

where D is the diagonal matrix of eigenvalues and $(\gamma_1, \gamma_2)^T$ are the coordinates of $(x, \, y)^T$ in the basis $\{v_1, \, v_2\}$, i.e.

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = S \begin{pmatrix} x \\ y \end{pmatrix}$$

Thus, we have ellipses associated with the level c described by the equation $6\gamma_1^2 + 4\gamma_2^2 = c$. For these eclipses major axis is along the line formed by v_1 and a minor axis is along the line formed by v_2. The ellipse crosses the major axis at $\pm\sqrt{c/4}$, and crosses the minor axis at $\pm\sqrt{c/6}$. Note, therefore, that the ellipse is longest in the direction of the eigenvector corresponding to the smallest eigenvalue. (Do you understand why this is? What is the value of c when the ellipse crosses the minor axis at ± 1?).

The form g is associated with M_g, which has one positive and one negative eigenvalue. This means that there is neither a maximum or a minimum at the origin. The maximum value of the form on the unit circle is 5, and the minimum value is -5. Level sets are given by $g(\gamma_1, \gamma_2) = 5\gamma_1^2 - 5\gamma_2^2 = c$, which generates a hyperbola. Rearranging this to $\gamma_1 = \pm\sqrt{\gamma_1^2 - c/5}$ informs us that the slopes of the asymptotes in the (γ_1, γ_2) plane are plus and minus one. Taking first the case that $c > 0$ and solving for $\gamma_2 = 0$, we see that the hyperbola crosses the γ_1-axis at $\gamma_1 = \pm\sqrt{c/5}$. For $c < 0$, we can no longer have $\gamma_2 = 0$, and we instead solve for $\gamma_1 = 0$ to learn that the hyperbola intersects the γ_2-axis at the points $\gamma_2 = \pm\sqrt{-c/5}$ (and for c = 0, the hyperbola reduces to the asymptotes).

5. Give a second-order Taylor approximation to the function

$$f(x, y, z) = \cos(x + y + z) - \cos x \cos y \cos z$$

assuming that x, y, z are small in absolute value. Estimate your approximation error.

Solution. Using second-order Taylor expansion for $\cos x$, we have

$$\begin{aligned}&\cos x \approx 1 - \frac{x^2}{2}, \quad \cos y \approx 1 - \frac{y^2}{2}, \quad \cos z \approx 1 - \frac{z^2}{2},\\ &\cos(x + y + z) \approx 1 - \frac{1}{2}(x + y + z)^2.\end{aligned}$$

Now, plugging this into our expression for f and ignoring expression that are smaller then second-order term we find

$$\begin{aligned}f(x, y, z) &= \cos(x + y + z) - \cos x \cos y \cos z\\ &\approx 1 - \frac{1}{2}(x + y + z)^2 - (1 - \frac{x^2}{2})(1 - \frac{y^2}{2})(1 - \frac{z^2}{2}) =\\ &= 1 - \frac{1}{2}(x^2 + y^2 + z^2) - (xy + xz + yz) - 1+\\ &\quad + \frac{1}{2}(x^2 + y^2 + z^2) - \frac{1}{4}(x^2y^2 + x^2z^2 + y^2z^2) + \frac{1}{8}x^2y^2z^2 =\\ &\quad - (xy + xz + yz).\end{aligned}$$

The error estimate is the difference between the true value of $f(x, y, z)$ and the Taylor expansion. We write this as:

$$E_3 = \cos(x + y + z) - \cos x \cos y \cos z + (xy + xz + yz)$$

Where the subscript 3 on E_3 indicates that this error term is a third order error term. It is also $O\left(\left|\begin{matrix}x\\y\\z\end{matrix}\right|^3\right)$.

6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable function. A point $x \in \mathbb{R}^n$ is a *critical point* of f if all the partial derivatives of f equal zero at x. A critical point is *nondegenerate* if the $n \times n$ matrix

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right)$$

is nonsignular.
Let x be a nondegenerate critical point of f. Prove that there is an open neighborhood of x which contains no other critical points (i.e. the nondegenerate critical points ar isolated).

Solution. Consider the following function $G : \mathbb{R}^n \to R$ given by $G(x) = ||\nabla f(x)||^2$. This function has a following Jackobian

$$DG(x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right).$$

By our assumptions, we have $G(x) = 0$, G is C^1, and $DG(x)$ nonsingular. Therefore, we can invoke Inverse Function Theorem which guarantees that G is locally injective in some neighborhood of a nondegenerate critical value x. In other words, there is no other solutions there.

7. Let $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$. Consider a mapping $f = (f_1, f_2)$ of $\mathbb{R}^5$ into $\mathbb{R}^2$ defined by the equation

$$f_1(\mathbf{x}, \mathbf{y}) = 2e^{x_1} + x_2 y_1 - 4y_2 + 3$$
$$f_2(\mathbf{x}, \mathbf{y}) = x_2 \cos x_1 - 6x_1 + 2y_1 - y_3$$

Let $\mathbf{x}^* = (0, 1)$, $\mathbf{y}^* = (3, 2, 7)$ and note that $f(\mathbf{x}^*, \mathbf{y}^*) = (0, 0)$. Use the implicit function theorem to prove that there is a C^1 mapping g, defined in a neighborhood of $\mathbf{y}^*$, such that $g(\mathbf{y}^*) = \mathbf{x}^*$ and $f(g(\mathbf{y}^*), \mathbf{y}^*) = (0, 0)$. Compute $Dg(\mathbf{y}^*)$.

Solution. Lets compute $Df(\mathbf{x}^*, \mathbf{y}^*)$. Calculating derivatives and plugging in values for $\mathbf{x}^*$ and $\mathbf{y}^*$ we obtain

$$Df(\mathbf{x}^*, \mathbf{y}^*) = \begin{bmatrix} 2 & 3 & 1 & -4 & 0 \\ -6 & 1 & 2 & 0 & -1 \end{bmatrix}.$$

Thus, we see that

$$D_x f(\mathbf{x}^*, \mathbf{y}^*) = \begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix}, \qquad D_y f(\mathbf{x}^*, \mathbf{y}^*) = \begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix}.$$

in particular, that $D_x f(\mathbf{x}^*, \mathbf{y}^*)$ is non-singular, and, thus, the Implicit Function Theorem guarantees the existence of C^1 mapping g, defined in the neighborhood of $\mathbf{y}^*$, such that $g(\mathbf{y}^*) = (0, 1)$ and $f(g(\mathbf{y}^*), \mathbf{y}^*) = 0$. Since

$$[D_x f^{-1}] = [D_x f]^{-1} = \frac{1}{20} \begin{bmatrix} 1 & -3 \\ 6 & 2 \end{bmatrix}$$

Using the Theorem to compute $Dg(\mathbf{y}^*)$ we obtain

$$Dg(\mathbf{y}^*) = -\frac{1}{20} \begin{bmatrix} 1 & -3 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/5 & -3/20 \\ -1/2 & 6/5 & 1/10 \end{bmatrix}$$