Economics 204
Fall 2010
Problem Set 5
Due Friday, August 13 in Lecture

1. Recall that a reflection across the x-axis can be achieved with the transformation $(x, y) \to (x, -y)$. Derive a transformation, T, which reflects a point across the line $y = 3x$.

   (a) First, calculate the action of T on the points $(1, 3)$ and $(-3, 1)$.

   (b) Next, write the matrix representation of T using these two vectors as bases.

   (c) Find S and S^{-1}, the matrices that changes coordinates under this basis to standard coordinates and back again.

   (d) Write the matrix representation of T in the standard basis.

   (e) Use point $(-3, 1)$ to verify the commutative diagram.

2. Similarity

   (a) Which matrices are similar to the identity matrix? to zero matrix?

   (b) What would your answers to (2a) suggest about similarity of the matrix of the form cI for some scalar c? Or, about a similarity of the diagonal matrix?

   (c) Show that if $T - \lambda I$ and N are similar matrices then T and $N + \lambda I$ are also similar.

3. Let A and B denote symmetric real $n \times n$ matrices, such that $AB = BA$. Prove that A and B have a common eigenvector in $\mathbb{R}^n$. Provide a counter-example showing the claim that *every* eigenvector of AB is also an eigenvector of BA is false (i.e. you can't strengthen the initial statement).

4. Consider the following quadratic forms:

$$f(x, y) = 5x^2 + 2xy + 5y^2,$$
$$g(x, y) = 10xy$$

   Answer the following questions for each of these forms:

   (a) Find a symmetric matrix M such that the form equals $[x\ y]\ M \begin{bmatrix} x \\ y \end{bmatrix}$.

   (b) Find the eigenvalues of the form.

   (c) Find an orthonormal basis of eigenvectors.

(d) Find a unitary matrix S such that $M = S^{-1}DS$, where D is a diagonal matrix.

(e) Describe the level sets of the form and state whether the form has a local maximum, local minimum, or neither at $(0, 0)$. (Level sets are solutions to $f(x, y) = c$ for some $c \in \mathbb{R}$.)

5. Give a second-order Taylor approximation to the function

$$f(x, y, z) = \cos(x + y + z) - \cos x \cos y \cos z$$

assuming that x, y, z are small in absolute value. Estimate your approximation error.

6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable function. A point $x \in \mathbb{R}^n$ is a *critical point* of f if all the partial derivatives of f equal zero at x. A critical point is *nondegenerate* if the $n \times n$ matrix

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right)$$

is nonsignular.
Let x be a nondegenerate critical point of f. Prove that there is an open neighborhood of x which contains no other critical points (i.e. the nondegenerate critical points ar isolated).

7. Let $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$. Consider a mapping $f = (f_1, f_2)$ of $\mathbb{R}^5$ into $\mathbb{R}^2$ defined by the equation

$$\begin{aligned} f_1(\mathbf{x}, \mathbf{y}) &= 2e^{x_1} + x_2 y_1 - 4y_2 + 3 \\ f_2(\mathbf{x}, \mathbf{y}) &= x_2 \cos x_1 - 6x_1 + 2y_1 - y_3 \end{aligned}$$

Let $\mathbf{x}^* = (0, 1)$, $\mathbf{y}^* = (3, 2, 7)$ and note that $f(\mathbf{x}^*, \mathbf{y}^*) = (0, 0)$. Use the implicit function theorem to prove that there is a C^1 mapping g, defined in a neighborhood of $\mathbf{y}^*$, such that $g(\mathbf{y}^*) = \mathbf{x}^*$ and $f(g(\mathbf{y}^*), \mathbf{y}^*) = (0, 0)$. Compute $Dg(\mathbf{y}^*)$.