Economics 204
Fall 2010
Problem Set 6 Suggested Solutions

1. Suppose $\Gamma : X \to 2^Y$ is a correspondence defined by $\Gamma(x) = \{f_1(x), ..., f_N(x)\}$, where $f_i : X \to Y$ is continuous for all $i \in \{1, 2, ..., N\}$. Prove that Γ is both lhc and uhc.

   Solution: To prove that Γ is lhc at $x_0 \in X$, we need to show that for every open set $O \subseteq Y$ with $\Gamma(x_0) \cap O \neq \varnothing$, there is an open set $U \subseteq X$ with $x_0 \in U$ such that $\forall x \in U : \Gamma(x) \cap O \neq \varnothing$.

   Let O be an open set in Y and let $\Gamma(x_0) \cap O \neq \varnothing$ for some $x_0 \in X$. This implies $f_i(x_0) \in O$ for some $i \in \{1, 2, ..., N\}$. Since f_i is continuous, $f_i^{-1}(O) \equiv U$ is open. We know $\forall x \in U : f_i(x) \in O$. But $f_i(x) \in \Gamma(x)$. Thus:

   $$\forall x \in U : \Gamma(x) \cap O \neq \varnothing.$$

   To prove that Γ is uhc at $x_0 \in X$, we need to show that for every open set $O \subseteq Y$ with $\Gamma(x_0) \subseteq O$, there is an open set $U \subseteq X$ with $x_0 \in U$ such that $\forall x \in U : \Gamma(x) \subseteq O$.

   Let O be an open set in Y and let $\Gamma(x_0) \subseteq O$ for some $x_0 \in X$. Consider:

   $$U = f_1^{-1}(O) \cap \ldots \cap f_N^{-1}(O).$$

   The set U is an intersection of finitely many open sets so it is open. Since clearly $x_0 \in U$, it is non-empty as well. Consider $\Gamma(x)$ for some $x \in U$. We have:

   $$\begin{aligned} \forall i : x \in f_i^{-1}(O) \quad &\Rightarrow \forall i : f_i(x) \in O \\ &\Rightarrow \Gamma(x) = \{f_1(x), \ldots, f_N(x)\} \subseteq O. \end{aligned}$$

2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function. Define $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

   $$F(x, \omega) = f(x) + \omega$$

   Show that there is a set $\Omega_0 \subset \mathbb{R}^n$ of Lebesgue measure zero such that, if $\omega \notin \Omega_0$, then for each x_0 satisfying $F(x_0, \omega_0) = 0$ there is an open set U containing x_0, an open set V containing ω_0, and a C^1 function $h : V \to U$ such that for all $\omega \in V$, $x = h(\omega)$ is the unique element of U satisfying $F(x, \omega) = 0$.

   Solution: If we can show that the Jacobian of F with respect to all of its arguments has full rank whenever $F(x, \omega) = 0$, then the Transversality theorem gives us that there is a set $\Omega_0 \subset \mathbb{R}^n$ of Lebesgue measure zero such that, if $\omega \notin \Omega_0$, then for each x_0 satisfying $F(x_0, \omega_0) = 0$, $D_x F(x_0, \omega_0)$ has full rank as well.

Taking the "full" Jacobian of F, and noting the Jacobian of f by Df,we have that:

$$DF(x, \omega) = [Df(x), I_n]$$

where I_n is the $n \times n$ identity matrix. This result comes from the fact that the Jacobian of F with respect to ω is the identity matrix ($\frac{\partial F_i}{\partial \omega_i} = 1$ and $\frac{\partial F_j}{\partial \omega_i} = 0$ $\forall j \neq i$).

I_n has rank n, so $DF(x, \omega)$ must also have rank n, so the condition of the Transversality theorem is satisfied. Thus, there is a set $\Omega_0 \subset \mathbb{R}^n$ of Lebesgue measure zero such that, if $\omega \notin \Omega_0$, then for each x_0 satisfying $F(x_0, \omega_0) = 0$, $|DF_x(x_0, \omega_0)| \neq 0$.

We now finish the problem by applying the implicit function theorem. This tells us directly that, whenever $F(x_0, \omega_0) = 0$ and $|DF_x(x_0, \omega_0)| \neq 0$, there is an open set U containing x_0, an open set V containing ω_0, and a C^1 function $h : V \to U$ such that for all $\omega \in V$,$x = h(\omega)$ is the unique element of U satisfying $F(x, \omega) = 0$.

3. Let $f : X \to X$ be continuous. Give an example of a set $X \subseteq \mathbb{R}^n$ and a continuous function f, such that f does not have a fixed point and X is:

   (a) closed, bounded, but not convex;

   (b) convex, closed, but not bounded;

   (c) convex, bounded, but not closed.

   Solution:

   (a) $X = \{0, 1\}$ and $f(x) = 1 - x$;

   (b) $X = [0, \infty)$ and $f(x) = x + 1$;

   (c) $X = (0, 1)$ and $f(x) = x^2$.

4. Let A be a nonempty, compact and convex subset of $\mathbb{R}^2$ such that if $(x, y) \in A$ for some $x, y \in \mathbb{R}$, then there exists some $z \in \mathbb{R}$ such that $(y, z) \in A$. Prove that $(x^*, x^*) \in A$ for some $x^* \in \mathbb{R}$. (Hint: Use Kakutani's Fixed Point Theorem)

   Solution: We will consider the correspondence $\Gamma : B \to 2^{\mathbb{R}}$ defined by $\Gamma(x) = \{y \in \mathbb{R} : (x, y) \in A\}$. where $B = \{x \in \mathbb{R}^n : (x, y) \in A \text{ for some } y \in A\}$ (intuitively, B is the set of first entries of the elements of A).

   Fix some $x \in B$. Then if $y \in \Gamma(x), (x, y) \in A$. But $(x, y) \in A \Rightarrow (y, z) \in A$ for some z. Hence $y \in B$ and thus $\Gamma(x) \subseteq B$ for all $x \in B$. This implies that Γ is a self-correspondence on B. Now notice that if we verify the conditions of KFPT, we will be done.

   We need to show that B is compact, convex and non-empty.

- B being nonempty follows directly from the definition of B and from A being nonempty.
- B is convex since A is convex. Namely for any $a_1, b_1 \in B, \exists a_2, b_2 \in \mathbb{R}$: $\{(a_1, a_2), (b_1, b_2)\} \subset A$:

$$\begin{aligned} \forall \alpha \in (0,1) \quad & : \alpha(a_1, a_2) + (1-\alpha)(b_1, b_2) \in A \\ & \Rightarrow \forall \alpha \in (0,1) : \alpha a_1 + (1-\alpha) b_1 \in B \\ & \Rightarrow B \text{ is convex.} \end{aligned}$$

- The projection function $d : \mathbb{R}^2 \to \mathbb{R}$ defined by $d(x, y) = x$ is continuous. Since B is the image of A under d (i.e. $B = d(A)$) and continuous functions preserve compactness, B is compact.

We need to show that Γ is convex-, closed-, and non-empty-valued, as well as uhc.

- Showing that Γ is non-empty-valued is trivial: it follows directly from the definitions of Γ and B.
- Let x be given and fix some $y, z \in \Gamma(x)$. This implies $(x, y), (x, z) \in A$. Since A is convex, $\alpha(x, y) + (1-\alpha)(x, z) = (x, \alpha y + (1-\alpha)z) \in A$ for all $\alpha \in [0, 1]$. Therefore $\alpha y + (1-\alpha)z \in \Gamma(x)$ for all $\alpha \in [0, 1]$ and Γ is convex-valued.
- Notice that the graph of Γ is simply the set A. Since A is closed and B is compact, we can use Theorem 11 (iii) from lecture 7 to deduce that Γ is closed-valued and uhc.

5. Let A and B be nonempty, convex subsets of $\mathbb{R}^n$ with $\text{int}A \neq \varnothing$. Using the Separating Hyperplane Theorem, prove that there exists $p \in \mathbb{R}^n$ with $p \neq 0$ such that $\sup p \cdot A \leq \inf p \cdot B$ if and only if $\text{int}A \cap B = \varnothing$. (Hint: You might want to use the result of Theorem 1.11 in de la Fuente, p.234.)

   Solution: Since $\text{int}A \cap B = \varnothing$, $\text{int}A$ and B can be separated ($\text{int}A$ is convex by the stated theorem). That is, there exists a non-zero vector $p \in \mathbb{R}^n$ such that:

$$\sup p \cdot (\text{int}A) \leq \inf p \cdot B.$$

   We can show that $\sup p \cdot (\text{int}A) = \sup p \cdot A$. It is obvious that $\sup p \cdot (\text{int}A) \leq \sup p \cdot A$. To show the other inequality, first we show that every element of a convex set with a non-empty interior is a limit of a sequence entirely contained in the interior[1]. Let $x \in A$ and $y \in \text{int}A \neq \varnothing$. Assume $B_\varepsilon(y) \subseteq \text{int}A$ for some $\varepsilon > 0$. Consider the sequence $\{z_n\} = \{\frac{1}{n}y + (1 - \frac{1}{n})x\}$. We examine $B_{\frac{\varepsilon}{n}}(z_n)$

[1] The following proof is identical to the proof of theorem 1.13 on p.235 in de la Fuente. In fact, you can use that theorem to get the required result.

for some n. Notice that for all $z' \in B_{\frac{\varepsilon}{n}}(z_n)$, $z' = z_n + h$, where $h \in \mathbb{R}^n$ and $||h|| < \frac{\varepsilon}{n}$. Now notice:

$$z' = z_n + h = \frac{1}{n}y + (1 - \frac{1}{n})x + \frac{1}{n}(nh) = (1 - \frac{1}{n})x + \frac{1}{n}(y + nh).$$

We know $||nh|| = n||h|| < n(\frac{\varepsilon}{n}) = \varepsilon$, therefore $y + nh \in B_\varepsilon(y) \subseteq A$. Hence z' is a convex combination of $x \in A$ and another element of A, so $z' \in A$. Since z' is an arbitrary element of $B_{\frac{\varepsilon}{n}}(z_n)$, we get $B_{\frac{\varepsilon}{n}}(z_n) \subseteq A$ and therefore $z_n \in \text{int}A$ for all n.

We constructed a sequence converging to an arbitrary $x \in A$ that is entirely contained in the interior of A and since $p \cdot x$ is continuous, we get:

$$\sup p \cdot A \leq \sup p \cdot (\text{int}A).$$

Conversely, suppose that A and B can be separated. That is, there exists $p \in \mathbb{R}^n$ such that:

$$\sup p \cdot A \leq c \leq \inf p \cdot B$$

Assume toward contradiction that $\text{int}A \cap B \neq \varnothing$. In particular, let $x \in \text{int}A \cap B$. Then $p \cdot x = c \geq \sup p \cdot A$. The vector p is not zero therefore it has at least one non-zero element. Without loss of generality, assume that is p_1 (i.e. the first element of p) and assume that $p_1 > 0$ (the other case is analogous). Since $x \in \text{int}A$, $\exists \varepsilon > 0 : x^* = x + (\varepsilon, 0, 0, \ldots, 0) \in A$. But then $p \cdot x^* > p \cdot x$ which is a contradiction.

6. Consider the second order linear differential equation given by $y'' = 4y' - 8y$.

   (a) Show how this equation can be rewritten as the following *first* order linear differential equation of two variables:

   $$\bar{y}'(t) = A\bar{y}(t)$$

   (b) Verbally describe the solutions of the first order system by analyzing the matrix A.

   (c) Solve the system when $y(0) = 3$ and $y'(0) = 7$.

Solution:

(a) Define the new variable $\bar{y} = \begin{bmatrix} y \\ y' \end{bmatrix}$. This gives us:

$$\bar{y}' = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ 4y' - 8y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = A\bar{y}.$$

(b) The eigenvalues of A are $2 \pm 2i$. Since the eigenvalues are complex, the solutions of the differential equation spiral around the origin. Since the real parts of the solutions are both positive, the system spirals outward and the solutions tend to infinity

(c) From the lecture notes, we immediately know the solution is of the form:

$$y(t) = e^{2t}(C\cos(2t) + D\sin(2t)).$$

Substituting in $t = 0$, we get $y(0) = C$. From the initial condition then, $C = 3$. So:

$$y(t) = 3e^{2t}\cos(2t) + De^{2t}\sin(2t).$$

Differentiating yields:

$$y'(t) = 6e^{2t}\cos(2t) - 6e^{2t}\sin(2t) + 2De^{2t}\sin(2t) + 2De^{2t}\cos(2t).$$

Setting $t = 0$ and using the initial condition for $y'(0)$ gives us $6 + 2D = 7$ so:

$$y(t) = 3e^{2t}\cos(2t) + \frac{1}{2}e^{2t}\sin(2t).$$