Econ 204 2011

Lecture 13

Outline

1. Fixed Points for Functions
2. Brouwer’s Fixed Point Theorem
3. Fixed Points for Correspondences
4. Kakutani’s Fixed Point Theorem
5. Separating Hyperplane Theorems

Announcements

- PS5 typo
- #2 (a) missing radius of convergence
- Comments about exam tomorrow at break
Fixed Points for Functions

**Definition 1.** Let $X$ be a nonempty set and $f : X \to X$. A point $x^* \in X$ is a fixed point of $f$ if $f(x^*) = x^*$.

$x^*$ is a fixed point of $f$ if it is “fixed” by the map $f$. 
The graph of the function $f(x)$ is shown in the Cartesian coordinate system. The graph is labeled with the notation $f(x)$ on the y-axis. Additionally, the line $x = y$ is included in the diagram to illustrate the symmetry of the function. The text "graph of $f$" is also noted on the graph.
Fixed Points for Functions

Examples:

1. Let $X = \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2x$. Then $x = 0$ is a fixed point of $f$ (and is the unique fixed point of $f$).

2. Let $X = \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x$. Then every point in $\mathbb{R}$ is a fixed point of $f$ (in particular, fixed points need not be unique).

3. Let $X = \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x + 1$. Then $f$ has no fixed points.
4. Let \( X = [0, 2] \) and \( f : X \to X \) be given by \( f(x) = \frac{1}{2}(x + 1) \). Then

\[
f(x) = \frac{1}{2}(x + 1) = x
\]

\[
\iff x + 1 = 2x
\]

\[
\iff x = 1
\]

So \( x = 1 \) is the unique fixed point of \( f \). Notice that \( f \) is a contraction (why?), so we already knew that \( f \) must have a unique fixed point on \( \mathbb{R} \) from the Contraction Mapping Theorem.

5. Let \( X = [0, \frac{1}{4}] \cup [\frac{3}{4}, 1] \) and \( f : X \to X \) be given by \( f(x) = 1 - x \). Then \( f \) has no fixed points.

\[
x = 1 - x \iff x = \frac{1}{2} \notin X
\]
6. Let $X = [-2, 2]$ and $f : X \to X$ be given by $f(x) = \frac{1}{2}x^2$. Then $f$ has two fixed points, $x = 0$ and $x = 2$. If instead $X' = (0, 2)$, then $f : X' \to X'$ but $f$ has no fixed points on $X'$.

7. Let $X = \{1, 2, 3\}$ and $f : X \to X$ be given by $f(1) = 2, f(2) = 3, f(3) = 1$ (so $f$ is a permutation of $X$). Then $f$ has no fixed points.

8. Let $X = [0, 2]$ and $f : X \to X$ be given by

$$f(x) = \begin{cases} 
    x + 1 & \text{if } x \leq 1 \\
    x - 1 & \text{if } x > 1
\end{cases}$$

Then $f$ has no fixed points.
A Simple Fixed Point Theorem

Theorem 1. Let $X = [a, b]$ for $a, b \in \mathbb{R}$ with $a < b$ and let $f : X \to X$ be continuous. Then $f$ has a fixed point.

Proof. Let $g : [a, b] \to \mathbb{R}$ be given by

$$g(x) = f(x) - x$$

If either $f(a) \neq a$ or $f(b) \neq b$, we’re done. So assume $f(a) > a$ and $f(b) < b$. Then

$$g(a) = f(a) - a > 0$$
$$g(b) = f(b) - b < 0$$

g is continuous, so by the Intermediate Value Theorem, $\exists x^* \in (a, b)$ such that $g(x^*) = 0$, that is, such that $f(x^*) = x^*$. \qed
$g(x) = f(x) - x$
discontinuous functions might not have fixed points
Brouwer’s Fixed Point Theorem

**Theorem 2** (Thm. 3.2. Brouwer’s Fixed Point Theorem). Let $X \subseteq \mathbb{R}^n$ be nonempty, compact, and convex, and let $f : X \to X$ be continuous. Then $f$ has a fixed point.
Sketch of Proof of Brouwer

Consider the case when the set $X$ is the unit ball in $\mathbb{R}^n$, i.e. $X = B_1[0] = B = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. Let $f : B \to B$ be a continuous function. Recall that $\partial B$ denotes the boundary of $B$, so $\partial B = \{x \in \mathbb{R}^n : \|x\| = 1\}$.

**Fact:** Let $B$ be the unit ball in $\mathbb{R}^n$. Then there is no continuous function $h : B \to \partial B$ such that $h(x') = x'$ for every $x' \in \partial B$.

See J. Franklin, Methods of Mathematical Economics, for an elementary (but long) proof.
There exists a function $h : B \to \partial B$ such that $h(x') = x'$ for all $x' \in \partial B$. 

$B$ is a set and $\partial B$ is its boundary. The function $h$ maps points on the boundary of $B$ back to the boundary.
Now to establish Brouwer's theorem, suppose, by way of contradiction, that $f$ has no fixed points in $B$. Thus for every $x \in B$, $x \neq f(x)$.

Since $x \neq f(x)$ for every $x$, we can carry out the following construction. For each $x \in B$, construct the line segment originating at $f(x)$ and going through $x$. Let $g(x)$ denote the intersection of this line segment with $\partial B$.

This construction is well-defined, and gives a continuous function $g : B \rightarrow \partial B$. Furthermore, if $x' \in \partial B$, then $x' = g(x')$. That is, $g|_{\partial B} = \text{id}_{\partial B}$. Since there are no such functions by the fact above, we have a contradiction. Therefore there exists $x^* \in B$ such that $f(x^*) = x^*$, that is, $f$ has a fixed point in $B$. 
\[ f(x) = x \]
Fixed Points for Correspondences

**Definition 2.** Let $X$ be nonempty and $\Psi : X \to 2^X$ be a correspondence. A point $x^* \in X$ is a fixed point of $\Psi$ if $x^* \in \Psi(x^*)$.

Note here that we do not require $\Psi(x^*) = \{x^*\}$, that is $\Psi$ need not be single-valued at $x^*$. So $x^*$ can be a fixed point of $\Psi$ but there may be other elements of $\Psi(x^*)$ different from $x^*$. 
Examples:

1. Let $X = [0, 4]$ and $\Psi : X \to 2^X$ be given by

$$\Psi(x) = \begin{cases} 
[x + 1, x + 2] & \text{if } x < 2 \\
[0, 4] & \text{if } x = 2 \\
[x - 2, x - 1] & \text{if } x > 2
\end{cases}$$

Then $x = 2$ is the unique fixed point of $\Psi$.

$$2 \in [0, 4] = \Psi(2)$$

2. Let $X = [0, 4]$ and $\Psi : X \to 2^X$ be given by

$$\Psi(x) = \begin{cases} 
[x + 1, x + 2] & \text{if } x < 2 \\
[0, 1] \cup [3, 4] & \text{if } x = 2 \\
[x - 2, x - 1] & \text{if } x > 2
\end{cases}$$

Then $\Psi$ has no fixed points.

$$2 \notin \Psi(2) = [0, 1] \cup [3, 4]$$
\[ y = x + 2 \]

\[ y = x - 1 \]

\[ x \in \mathcal{H}(a) = [0, 4] \]
\[ V \text{ open if } \psi(x) \subseteq V \wedge U \supseteq 2 \text{ open } \]

\[ \forall x \in U \]

\[ \psi(x) \subseteq V \]

Note: \( \psi \) works in both cases.
Kakutani’s Fixed Point Theorem

Theorem 3. (Thm. 3.4’. Kakutani’s Fixed Point Theorem) Let $X \subseteq \mathbb{R}^n$ be a non-empty, compact, convex set and $\Psi : X \to 2^X$ be an upper hemi-continuous correspondence with non-empty, convex, compact values. Then $\Psi$ has a fixed point in $X$.

Proof. (sketch) Here, the idea is to use Brouwer’s theorem after appropriately approximating the correspondence with a function. The catch is that there won’t necessarily exist a continuous selection from $\Psi$, that is, a continuous function $f : X \to X$ such that $f(x) \in \Psi(x)$ for every $x \in X$. If such a function existed, then by applying Brouwer to $f$ we would have a fixed point of $\Psi$ (because if $\exists x^* \in X$ such that $x^* = f(x^*)$, then $x^* = f(x^*) \in \Psi(x^*)$).
\[ f(x) \in \Psi(x) \quad \text{and } f \text{ is continuous} \]

\[ \Psi(x) \text{ is a convex set} \]
Instead, we look for a weaker type of approximation. Let $X \subset \mathbb{R}^n$ be a non-empty, compact, convex set, and let $\Psi : X \to 2^X$ be an uhc correspondence with non-empty, compact, convex values. For every $\varepsilon > 0$, define the $\varepsilon$ ball about graph $\Psi$ to be

$$B_{\varepsilon}(\text{graph } \Psi) = \left\{ z \in X \times X : d(z, \text{graph } \Psi) = \inf_{(x,y) \in \text{graph } \Psi} d(z, (x,y)) < \varepsilon \right\}$$

Here $d$ denotes the ordinary Euclidean distance in $\mathbb{R}^n$. If $\Psi$ is an uhc correspondence, then for every $\varepsilon > 0$ there exists a continuous function $f_\varepsilon : X \to X$ such that $\text{graph } f_\varepsilon \subseteq B_{\varepsilon}(\text{graph } \Psi)$. 
Now by letting $\varepsilon \to 0$, this means that we can find a sequence of continuous functions $\{f_n\}$ such that $\text{graph } f_n \subseteq B_{\frac{1}{n}}(\text{graph } \Psi)$ for each $n$. By Brouwer’s Fixed Point Theorem, each function $f_n$ has a fixed point $\hat{x}_n \in X$, and

$$(\hat{x}_n, \hat{x}_n) = (\hat{x}_n, f_n(\hat{x}_n)) \in \text{graph } f_n \subseteq B_{\frac{1}{n}}(\text{graph } \Psi) \text{ for each } n$$

So for each $n$ there exists $(x_n, y_n) \in \text{graph } \Psi$ such that

$$d(\hat{x}_n, x_n) < \frac{1}{n} \text{ and } d(\hat{x}_n, y_n) < \frac{1}{n}$$

Since $X$ is compact, $\{\hat{x}_n\}$ has a convergent subsequence $\{\hat{x}_{n_k}\}$, with $\hat{x}_{n_k} \to \hat{x} \in X$. Then $x_{n_k} \to \hat{x}$ and $y_{n_k} \to \hat{x}$. Since $\Psi$ is uhc and closed-valued, it has closed graph, so $(\hat{x}, \hat{x}) \in \text{graph } \Psi$. Thus $\hat{x} \in \Psi(\hat{x})$, that is, $\hat{x}$ is a fixed point of $\Psi$. $\square$
Separating Hyperplane Theorems

**Theorem 4** (1.26, Separating Hyperplane Theorem). Let $A, B \subseteq \mathbb{R}^n$ be nonempty, disjoint convex sets. Then there exists a nonzero vector $p \in \mathbb{R}^n$ such that

$$p \cdot a \leq p \cdot b \quad \forall a \in A, b \in B$$
$H = \{ x \in \mathbb{R}^n : p \cdot x = c \}$

$p \cdot a \leq c \quad \forall a \in A$

$p \cdot b \geq c \quad \forall b \in B$
\[ \exists \mathbf{p} \in \mathbb{R}^n \quad \mathbf{p} - \mathbf{a} \leq \mathbf{p} - \mathbf{b} \]

will need convexity: no hyperplane separates
\[ A \setminus B \]
Separating a Point from a Set

**Theorem 5.** Let $Y \subseteq \mathbb{R}^n$ be a nonempty convex set and $x \notin Y$. Then there exists a nonzero vector $p \in \mathbb{R}^n$ such that

$$p \cdot x \leq p \cdot y \quad \forall y \in Y$$

**Proof.** We sketch the proof in the special case that $Y$ is compact. We will see that in this case we actually get a stronger conclusion:

$$\exists p \in \mathbb{R}^n, p \neq 0 \text{ s.t. } p \cdot x < p \cdot y \quad \forall y \in Y$$

Choose $y_0 \in Y$ such that $|y_0 - x| = \inf\{|y - x| : y \in Y\}$; such a point exists because $Y$ is compact, so the distance function $g(y) = |y - x|$ assumes its minimum on $Y$. Since $x \notin Y$, $x \neq y_0$, so $y_0 - x \neq 0$. Let $p = y_0 - x$. The set

$$H = \{z \in \mathbb{R}^n : p \cdot z = p \cdot y_0\}$$

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is the hyperplane perpendicular to $p$ through $y_0$. (See Figure 12.)

Then

$$p \cdot y_0 = (y_0 - x) \cdot y_0$$
$$= (y_0 - x) \cdot (y_0 - x) + x$$
$$= (y_0 - x) \cdot (y_0 - x) + (y_0 - x) \cdot x$$
$$= |y_0 - x|^2 + p \cdot x$$
$$> p \cdot x$$

We claim that

$$y \in Y \Rightarrow p \cdot y \geq p \cdot y_0 > p \cdot x$$

If not, suppose there exists $y \in Y$ such that $p \cdot y < p \cdot y_0$. Given $\alpha \in (0, 1)$, let

$$w_\alpha = \alpha y + (1 - \alpha)y_0$$
Since $Y$ is convex, $w_\alpha \in Y$. Then for $\alpha$ sufficiently close to zero,

$$
\begin{align*}
|x - w_\alpha|^2 &= |x - \alpha y - (1 - \alpha)y_0|^2 \\
&= |x - y_0 + \alpha(y_0 - y)|^2 \\
&= |-p + \alpha(y_0 - y)|^2 \\
&= |p|^2 - 2\alpha p \cdot (y_0 - y) + \alpha^2|y_0 - y|^2 \\
&= |p|^2 + \alpha \left( -2p \cdot (y_0 - y) + \alpha|y_0 - y|^2 \right) \\
&< |p|^2 \quad \text{for} \ \alpha \ \text{close to} \ 0, \ \text{as} \ p \cdot y_0 > p \cdot y
\end{align*}
$$

Thus for $\alpha$ sufficiently close to zero,

$$
|w_\alpha - x| < |y_0 - x|
$$

which implies $y_0$ is not the closest point in $Y$ to $x$, contradiction.
\[ H = \{ z : p \cdot z = p \cdot y_0 \} \]

? \{ y \in Y : p \cdot y < p \cdot y_0 \}?
The general version of the Separating Hyperplane Theorem can be derived from this special case by noting that if $A \cap B = \emptyset$, then $0 \notin A - B = \{a - b : a \in A, b \in B\}$. 
Strict Separation

For the special case of $Y$ compact and $X = \{x\}$, we actually could strictly separate $Y$ and $X$:

\[ \times \not\leftrightarrow \exists p \in \mathbb{R}^n, p \neq 0 \text{ s.t. } p \cdot x < p \cdot y \quad \forall y \in Y \]

When can we do this in general? Will require additional assumptions...
A, B nonempty, disjoint, convex.

\[ \exists \exists \, p \text{ s.t. } p \cdot a = p \cdot b \land \exists \in A \land \exists \in B \]

\[ p \cdot \overrightarrow{a} = p \cdot \overrightarrow{b} \] some \[ \overrightarrow{a} \in A \land \overrightarrow{b} \in B \]
Strict Separation

Theorem 6. (Strict Separating Hyperplane Theorem) Let $A, B \subseteq \mathbb{R}^n$ be nonempty, disjoint, closed, convex sets. Then there exists a nonzero vector $p \in \mathbb{R}^n$ such that

$$p \cdot a < p \cdot b \quad \forall a \in A, b \in B$$