Econ 204 2011
Lecture 3

Outline

1. Metric Spaces and Normed Spaces
2. Convergence of Sequences in Metric Spaces
3. Sequences in \( \mathbb{R} \) and \( \mathbb{R}^n \)
Metric Spaces and Metrics

Generalize distance and length notions in $\mathbb{R}^n$

**Definition 1.** A metric space is a pair $(X, d)$, where $X$ is a set and $d : X \times X \to \mathbb{R}_+$ a function satisfying

\[ \mathbb{R}_+ := \{ r \in \mathbb{R} : r \geq 0 \} \]

1. $d(x, y) \geq 0$, $d(x, y) = 0 \Leftrightarrow x = y \ \forall x, y \in X$

2. $d(x, y) = d(y, x) \ \forall x, y \in X$

3. triangle inequality:

\[ d(x, z) \leq d(x, y) + d(y, z) \ \forall x, y, z \in X \]
A function $d : X \times X \to \mathbb{R}_+$ satisfying 1-3 above is called a metric on $X$.

A metric gives a notion of distance between elements of $X$. 
Normed Spaces and Norms

Definition 2. Let $V$ be a vector space over $\mathbb{R}$. A norm on $V$ is a function $\| \cdot \| : V \rightarrow \mathbb{R}_+$ satisfying

1. $\|x\| \geq 0 \ \forall x \in V$

2. $\|x\| = 0 \iff x = 0 \ \forall x \in V$

3. triangle inequality:

   $$\|x + y\| \leq \|x\| + \|y\| \ \forall x, y \in V$$
A normed vector space is a vector space over $\mathbb{R}$ equipped with a norm.

A norm gives a notion of length of a vector in $V$. 

$$4. \, \|\alpha x\| = |\alpha||x| \quad \forall \alpha \in \mathbb{R}, x \in V$$
Normed Spaces and Norms

Example: In $\mathbb{R}^n$, standard notion of distance between two vectors $x$ and $y$ measures length of difference $x - y$, i.e.,

$$d(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}.$$ 

In an abstract normed vector space, the norm can be used analogously to define a notion of distance.

**Theorem 1.** Let $(V, \| \cdot \|)$ be a normed vector space. Let $d : V \times V \Rightarrow \mathbb{R}_+$ be defined by

$$d(v, w) = \|v - w\|$$

Then $(V, d)$ is a metric space.
Proof. We must verify that $d$ satisfies all the properties of a metric.

1. Let $v, w \in V$. Then by definition, $d(v, w) = \|v - w\| \geq 0$ (why?), and

$$d(v, w) = 0 \iff \|v - w\| = 0 \iff v - w = 0\text{ additive id in } V \iff (v + (-w)) + w = w \iff v + ((-w) + w) = w \iff v + 0 = w \iff v = w$$

2. First, note that for any $x \in V$, $0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x$, so $0 \cdot x = 0$. Then $0 = 0 \cdot x = (1 - 1) \cdot x = 1 \cdot x + (-1) \cdot x = 0 \text{ add id in } V$
\[ x + (-1) \cdot x, \text{ so we have } (-1) \cdot x = (-x). \text{ Then let } v, w \in V. \]

\[
d(v, w) = \|v - w\| \\
= | -1| \|v - w\| \\
= \|( -1)(v + (-w))\| \\
= \|( -1)v + (-1)(-w)\| \\
= \| - v + w\| \\
= \|w + (-v)\| \\
= \|w - v\| \\
= d(w, v)
\]
Let $u, w, v \in V$.

$$d(u, w) = \|u - w\| \leq 0$$
$$= \|u + (v - v) - w\|$$
$$= \|u - v + (v - w)\|$$
$$\leq \|u - v\| + \|v - w\|$$
$$= d(u, v) + d(v, w)$$

Thus $d$ is a metric on $V$.  

\[\square\]
Normed Spaces and Norms

Examples

• $\mathbb{E}^n$: $n$-dimensional Euclidean space.

$$V = \mathbb{R}^n, \quad \|x\|_2 = |x| = \sqrt{\sum_{i=1}^{n} (x_i)^2}$$

$$x = (x_1, \ldots, x_n)$$

• $V = \mathbb{R}^n, \quad \|x\|_1 = \sum_{i=1}^{n} |x_i|$ (the “taxi cab” norm or $L^1$ norm)

• $V = \mathbb{R}^n, \quad \|x\|_{\infty} = \max\{|x_1|, \ldots, |x_n|\}$ (the maximum norm, or sup norm, or $L^\infty$ norm)
$C([0,1]) : f: [0,1] \rightarrow \mathbb{R}$, continuous

$\forall$
- $C([0,1])$, $\|f\|_{\infty} = \sup \{|f(t)| : t \in [0,1]\}$

$\forall$
- $C([0,1])$, $\|f\|_{2} = \sqrt{\int_{0}^{1} (f(t))^2 \, dt}$

$\forall$
- $C([0,1])$, $\|f\|_{1} = \int_{0}^{1} |f(t)| \, dt$
Normed Spaces and Norms

**Theorem 2 (Cauchy-Schwarz Inequality).**

If \( v, w \in \mathbb{R}^n \), then

\[
\left( \sum_{i=1}^{n} v_i w_i \right)^2 \leq \left( \sum_{i=1}^{n} v_i^2 \right) \left( \sum_{i=1}^{n} w_i^2 \right)
\]

\[
|v \cdot w|^2 \leq |v|^2 |w|^2 = \|v\|_2 \|w\|_2
\]

\[
|v \cdot w| \leq |v| \|w\| = \|v\|_2 \|w\|_2
\]

- Learn some proof
- Triangle inequality of standard norm \( \| \cdot \|_2 \) in the follows from C.-S. - nice exercise
Equivalent Norms

A given vector space may have many different norms: if $\| \cdot \|$ is a norm on a vector space $V$, so are $2\| \cdot \|$ and $3\| \cdot \|$ and $k\| \cdot \|$ for any $k > 0$.

Less trivially, $\mathbb{R}^n$ supports many different norms as in the examples above. Different norms on a given vector space yield different geometric properties.
Look at \( \{ x \in \mathbb{R}^2 : \|x\| = 1 \} \) for different choices of \( \|\cdot\| \).

- **Standard norm**
  \( \{ x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1 \} \)

- **Sup norm**
  \( \{ x \in \mathbb{R}^2 : \max(\|x_1\|, \|x_2\|) = 1 \} \)

- **L^1 norm**
  \( \{ x \in \mathbb{R}^2 : |x_1| + |x_2| = 1 \} \)

Unit balls around 0 in different norms.
Equivalent Norms

Definition 3. Two norms $\| \cdot \|$ and $\| \cdot \|^*$ on the same vector space $V$ are said to be Lipschitz-equivalent (or equivalent) if

$\exists m, M > 0$ s.t. $\forall x \in V, \quad m\|x\| \leq \|x\|^* \leq M\|x\|

Equivalently, $\exists m, M > 0$ s.t. $\forall x \in V, x \neq 0, \quad m \leq \frac{\|x\|^*}{\|x\|} \leq M$

\[ \small{\textbf{This is an equivalence relation (exercise)}} \]
Equivalent Norms

If two norms are equivalent, then they define the same notions of convergence and continuity. For topological purposes, equivalent norms are indistinguishable.

For example, suppose two norms \( \| \cdot \| \) and \( \| \cdot \|^{*} \) on the vector space \( V \) are equivalent, and fix \( x \in V \). Let \( \varepsilon > 0 \)

\[
B_{\varepsilon}(x, \| \cdot \|) = \{ y \in V : \| x - y \| < \varepsilon \}
\]

\[
B_{\varepsilon}(x, \| \cdot \|^{*}) = \{ y \in V : \| x - y \|^{*} < \varepsilon \}
\]

Then for any \( \varepsilon > 0 \),

\[
B_{\varepsilon}(x, \| \cdot \|) \subseteq B_{\varepsilon}(x, \| \cdot \|^{*}) \subseteq B_{\varepsilon}(x, \| \cdot \|)
\]
norms on $\mathbb{R}^n$ are equivalent
Equivalent Norms

In $\mathbb{R}^n$ (or any finite-dimensional normed vector space), all norms are equivalent. Roughly, up to a difference in scaling, for topological purposes there is a unique norm in $\mathbb{R}^n$.

**Theorem 3.** All norms on $\mathbb{R}^n$ are equivalent.

Infinite-dimensional spaces support norms that are not equivalent. For example, on $C([0,1])$, let $f_n$ be the function

$$f_n(t) = \begin{cases} 1 - nt & \text{if } t \in [0, \frac{1}{n}] \\ 0 & \text{if } t \in (\frac{1}{n}, 1] \end{cases}$$

Then

$$\frac{\|f_n\|_1}{\|f_n\|_\infty} = \frac{1}{2n} = \frac{1}{2n} \to 0$$

$$\frac{1}{2n} = \|f_n\|_1 = \int_{0}^{1} |f_n(t)| \, dt = \int_{0}^{\frac{1}{n}} (1-nt) \, dt$$
Definition 4. In a metric space \((X, d)\), a subset \(S \subseteq X\) is bounded if \(\exists x \in X, \beta \in \mathbb{R}\) such that \(\forall s \in S, d(s, x) \leq \beta\).

In a metric space \((X, d)\), define

\[ B_\varepsilon(x) = \{ y \in X : d(y, x) < \varepsilon \} \]

= "open ball with center \(x\) and radius \(\varepsilon\)

\[ B_\varepsilon[x] = \{ y \in X : d(y, x) \leq \varepsilon \} \]

= "closed ball with center \(x\) and radius \(\varepsilon\)"
Metrics and Sets

We can use the metric $d$ to define a generalization of "radius". In a metric space $(X, d)$, define the \textit{diameter} of a subset $S \subseteq X$ by

$$\text{diam } (S) = \sup \{d(s, s') : s, s' \in S\}$$

Similarly, we can define the distance from a point to a set, and distance between sets, as follows:

$$d(A, x) = \inf_{a \in A} d(a, x)$$
$$d(A, B) = \inf_{a \in A} d(B, a)$$
$$= \inf \{d(a, b) : a \in A, b \in B\}$$

But $d(A, B)$ is \textbf{not} a metric. (why ??)
Convergence of Sequences

**Definition 5.** Let \((X, d)\) be a metric space. A sequence \(\{x_n\}\) converges to \(x\) (written \(x_n \to x\) or \(\lim_{n \to \infty} x_n = x\)) if

\[
\forall \varepsilon > 0 \ \exists N(\varepsilon) \in \mathbb{N} \ s.t. \ n > N(\varepsilon) \implies d(x_n, x) < \varepsilon
\]

Notice that this is exactly the same as the definition of convergence of a sequence of real numbers, except we replace the standard measure of distance \(|\cdot|\) in \(\mathbb{R}\) by the general metric \(d\).
\[ \exists \epsilon > 0 \]
\[ B_{\epsilon}(x) \]
\[ \exists N(\epsilon) \text{ s.t. } \]
\[ \forall n \geq N(\epsilon) \Rightarrow x_n \in B_{\epsilon}(x) \]
\[ \{x_n\} \]
Uniqueness of Limits

**Theorem 4** (Uniqueness of Limits). *In a metric space $(X, d)$, if $x_n \to x$ and $x_n \to x'$, then $x = x'$."

**Proof.** Suppose $\{x_n\}$ is a sequence in $X$, $x_n \to x$, $x_n \to x'$, $x \neq x'$.
Since $x \neq x'$, $d(x, x') > 0$. Let

$$
\varepsilon = \frac{d(x, x')}{2}
$$

Then there exist $N(\varepsilon)$ and $N'(\varepsilon)$ such that

$$
n > N(\varepsilon) \Rightarrow d(x_n, x) < \varepsilon \quad \text{and} \quad x_n \to x
$$

$$
n > N'(\varepsilon) \Rightarrow d(x_n, x') < \varepsilon \quad \text{and} \quad x_n \to x'
$$

Choose

$$
n > \max\{N(\varepsilon), N'(\varepsilon)\}
$$
Then

\[ d(x, x') \leq d(x, x_n) + d(x_n, x') \]
\[ < \varepsilon + \varepsilon \]
\[ = 2\varepsilon \]
\[ = d(x, x') \]

\[ d(x, x') < d(x, x') \]

a contradiction.
\[ d(x, x') = \frac{d(x, x')}{2} \]
Cluster Points

**Definition 6.** An element \( c \) is a cluster point of a sequence \( \{x_n\} \) in a metric space \((X, d)\) if \( \forall \varepsilon > 0, \{n : x_n \in B_\varepsilon(c)\} \) is an infinite set. Equivalently,

\[
\forall \varepsilon > 0, N \in \mathbb{N} \ \exists n > N \text{ s.t. } x_n \in B_\varepsilon(c)
\]

\( \{x_n\} \) arbitrarily close to \( c \) infinitely often

**Example:**

\[ x_n = \begin{cases} 
1 - \frac{1}{n} & \text{if } n \text{ even} \\
\frac{1}{n} & \text{if } n \text{ odd}
\end{cases} \]

For \( n \) large and odd, \( x_n \) is close to zero; for \( n \) large and even, \( x_n \) is close to one. The sequence does not converge; the set of cluster points is \( \{0, 1\} \).
Subsequences

If \( \{x_n\} \) is a sequence and \( n_1 < n_2 < n_3 < \cdots \) then \( \{x_{n_k}\} \) is called a subsequence.

Note that a subsequence is formed by taking some of the elements of the parent sequence, in the same order.

Example: \( x_n = \frac{1}{n} \), so \( \{x_n\} = (1, \frac{1}{2}, \frac{1}{3}, \ldots) \). If \( n_k = 2k \), then \( \{x_{n_k}\} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots) \).
Cluster Points and Subsequences

**Theorem 5** (2.4 in De La Fuente, plus ...). Let \((X, d)\) be a metric space, \(c \in X\), and \(\{x_n\}\) a sequence in \(X\). Then \(c\) is a cluster point of \(\{x_n\}\) if and only if there is a subsequence \(\{x_{n_k}\}\) such that \(\lim_{k \to \infty} x_{n_k} = c\).

**Proof.** Suppose \(c\) is a cluster point of \(\{x_n\}\). We inductively construct a subsequence that converges to \(c\). For \(k = 1\), \(\{n : x_n \in B_1(c)\}\) is infinite, so nonempty; let

\[
n_1 = \min\{n : x_n \in B_1(c)\}
\]

Now, suppose we have chosen \(n_1 < n_2 < \cdots < n_k\) such that

\[
x_{n_j} \in B_{\frac{1}{j}}(c) \text{ for } j = 1, \ldots, k
\]
\{n : x_n \in B^{1 \frac{1}{k+1}}_1(c)\} is infinite, so it contains at least one element bigger than \(n_k\), so let

\[ n_{k+1} = \min \left\{ n : n > n_k, x_n \in B^{1 \frac{1}{k+1}}_1(c) \right\} \]

Thus, we have chosen \(n_1 < n_2 < \cdots < n_k < n_{k+1}\) such that

\[ x_{n_j} \in B^{1 \frac{1}{j}}_1(c) \text{ for } j = 1, \ldots, k, k+1 \]

Thus, by induction, we obtain a subsequence \(\{x_{n_k}\}\) such that

\[ x_{n_k} \in B^{1 \frac{1}{k}}_1(c) \]

Given any \(\varepsilon > 0\), by the Archimedean property, there exists \(N(\varepsilon) > 1/\varepsilon\). (why?)

\[ k > N(\varepsilon) \Rightarrow x_{n_k} \in B^{1 \frac{1}{k}}_1(c) \quad \frac{1}{k} < \varepsilon \]

\[ \Rightarrow x_{n_k} \in B_\varepsilon(c) \]
so

\[ x_{n_k} \to c \text{ as } k \to \infty \]

Conversely, suppose that there is a subsequence \( \{x_{n_k}\} \) converging to \( c \). Given any \( \varepsilon > 0 \), there exists \( K \in \mathbb{N} \) such that

\[ k > K \Rightarrow d(x_{n_k}, c) < \varepsilon \Rightarrow x_{n_k} \in B_{\varepsilon}(c) \]

Therefore,

\[ \{n : x_n \in B_{\varepsilon}(c)\} \supseteq \{n_{K+1}, n_{K+2}, n_{K+3}, \ldots\} \]

Since \( n_{K+1} < n_{K+2} < n_{K+3} < \cdots \), this set is infinite, so \( c \) is a cluster point of \( \{x_n\} \). \qed
Sequences in $\mathbb{R}$ and $\mathbb{R}^m$

**Definition 7.** A sequence of real numbers $\{x_n\}$ is increasing (decreasing) if $x_{n+1} \geq x_n$ ($x_{n+1} \leq x_n$) for all $n$.

**Definition 8.** If $\{x_n\}$ is a sequence of real numbers, $\{x_n\}$ tends to infinity (written $x_n \to \infty$ or $\lim x_n = \infty$) if

$$\forall K \in \mathbb{R} \exists N(K) \text{ s.t. } n > N(K) \Rightarrow x_n > K$$

Similarly define $x_n \to -\infty$ or $\lim x_n = -\infty$. 
Increasing and Decreasing Sequences

**Theorem 6** (Theorem 3.1’). Let \( \{x_n\} \) be an increasing (decreasing) sequence of real numbers. Then

\[
\lim_{n \to \infty} x_n = \sup \{x_n : n \in \mathbb{N}\}
\]

\[
(\lim_{n \to \infty} x_n = \inf \{x_n : n \in \mathbb{N}\})
\]

In particular, the limit exists.

Read diff proof, think about how to handle the unbounded case
Lim Sups and Lim Infs

Consider a sequence \( \{x_n\} \) of real numbers. Let

\[
\alpha_n = \sup\{x_k : k \geq n\} = \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\}
\]

\[
\beta_n = \inf\{x_k : k \geq n\} = \inf\{x_n, x_{n+1}, x_{n+2}, \ldots\}
\]

Either \( \alpha_n = +\infty \) for all \( n \), or \( \alpha_n \in \mathbb{R} \) and \( \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots \).

Either \( \beta_n = -\infty \) for all \( n \), or \( \beta_n \in \mathbb{R} \) and \( \beta_1 \leq \beta_2 \leq \beta_3 \leq \cdots \).
**Definition 9.**

\[
\limsup_{n \to \infty} x_n = \begin{cases} 
  +\infty & \text{if } \alpha_n = +\infty \text{ for all } n \\
  \lim \alpha_n & \text{otherwise.}
\end{cases}
\]

\[
\liminf_{n \to \infty} x_n = \begin{cases} 
  -\infty & \text{if } \beta_n = -\infty \text{ for all } n \\
  \lim \beta_n & \text{otherwise.}
\end{cases}
\]

**Theorem 7.** Let \( \{x_n\} \) be a sequence of real numbers. Then

\[
\lim_{n \to \infty} x_n = \gamma \in \mathbb{R} \cup \{-\infty, \infty\} \\
\iff \limsup_{n \to \infty} x_n = \liminf_{n \to \infty} x_n = \gamma
\]
Increasing and Decreasing Subsequences

**Theorem 8** (Theorem 3.2, Rising Sun Lemma). Every sequence of real numbers contains an increasing subsequence or a decreasing subsequence or both.
Proof. Let

\[ S = \{ s \in \mathbb{N} : x_s > x_n \ \forall n > s \} \]

Either \( S \) is infinite, or \( S \) is finite.

If \( S \) is infinite, let

\[
\begin{align*}
  n_1 &= \min S & \text{first time} \\
  n_2 &= \min (S \setminus \{n_1\}) & \text{next time} \\
  n_3 &= \min (S \setminus \{n_1, n_2\}) & \text{next next time} \\
  & \vdots \\
  n_{k+1} &= \min (S \setminus \{n_1, n_2, \ldots, n_k\})
\end{align*}
\]
Then \( n_1 < n_2 < n_3 < \cdots \).

\[
\begin{align*}
\quad & x_{n_1} > x_{n_2} \quad \text{since } n_1 \in S \text{ and } n_2 > n_1 \\
\quad & x_{n_2} > x_{n_3} \quad \text{since } n_2 \in S \text{ and } n_3 > n_2 \\
\quad & \vdots \\
\quad & x_{n_k} > x_{n_{k+1}} \quad \text{since } n_k \in S \text{ and } n_{k+1} > n_k \\
\quad & \vdots
\end{align*}
\]

so \( \{x_{n_k}\} \) is a strictly decreasing subsequence of \( \{x_n\} \).

If \( S \) is finite and nonempty, let \( n_1 = (\max S) + 1 \); if \( S = \emptyset \), let \( n_1 = 1 \). Then

\[
\begin{align*}
\quad & n_1 \notin S \quad \text{so } \exists n_2 > n_1 \text{ s.t. } x_{n_2} \geq x_{n_1} \\
\quad & n_2 \notin S \quad \text{so } \exists n_3 > n_2 \text{ s.t. } x_{n_3} \geq x_{n_2} \\
\quad & \vdots \\
\quad & n_k \notin S \quad \text{so } \exists n_{k+1} > n_k \text{ s.t. } x_{n_{k+1}} \geq x_{n_k} \\
\quad & \vdots
\end{align*}
\]
so $\{x_{n_k}\}$ is a (weakly) increasing subsequence of $\{x_n\}$. 

\qed
Bolzano-Weierstrass Theorem

**Theorem 9** (Thm. 3.3, Bolzano-Weierstrass). *Every bounded sequence of real numbers contains a convergent subsequence.*

**Proof.** Let \( \{x_n\} \) be a bounded sequence of real numbers. By the Rising Sun Lemma, find an increasing or decreasing subsequence \( \{x_{n_k}\} \). If \( \{x_{n_k}\} \) is increasing, then by Theorem 3.1',

\[
\lim x_{n_k} = \sup \{x_{n_k} : k \in \mathbb{N}\} \leq \sup \{x_n : n \in \mathbb{N}\} < \infty
\]

since the sequence is bounded; since the limit is finite, the subsequence converges. Similarly, if the subsequence is decreasing, it converges. \( \square \)