1. (a) Prove that \( y = h^3 \) is both \( o(|h|^2) \) as \( h \to 0 \) and \( O(|h|^3) \) as \( h \to 0 \).

   (b) Prove that \( y = \sin(h) \) is not \( o(|h|) \) as \( h \to 0 \) but is \( O(|h|) \) as \( h \to 0 \). (You can use the fact that \( |\sin(h)| \leq |h| \)).

2. (a) Prove that the following identity holds for \(-1 < x \leq 1\):

   \[
   \ln(x + 1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n.
   \]

   (b) Find the second-order Taylor expansion of:

   \[
   f(x, y) = -x^2 + 2xy + 3y^2 - 6x - 2y - 4
   \]

   around \( (x, y) = (-\pi/4, \ln 42) \).

   (c) Find the second-order Taylor expansion of \( g(x, y) = y^x \) around \( (x, y) = (1, 1) \).

3. Define \( f : \mathbb{R}^3 \to \mathbb{R} \) by

   \[
   f(x, y, z) = x^2 y + e^x + z.
   \]

   Show that there exists a differentiable function \( g \) in some neighborhood of \( (1, -1) \) in \( \mathbb{R}^2 \), such that \( g(1, -1) = 0 \) and

   \[
   f(g(y, z), y, z) = 0.
   \]

   Compute \( Dg(1, -1) \).

4. Let \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by \( F(x, y) = (e^y \cos(x), e^y \sin(x)) \).

   (a) Show that \( F \) satisfies the prerequisites of the Inverse Function Theorem for all \( (x, y) \in \mathbb{R}^2 \) (and is therefore locally injective everywhere) but \( F \) is not globally injective.
(b) Compute the Jacobian of the local inverse of $F$ and evaluate it at $F(\frac{\pi}{3}, 0)$.

(c) Find an explicit formula for the continuous inverse of $F$ mapping a neighborhood of $F(\frac{\pi}{3}, 0)$ into a neighborhood of $(\frac{\pi}{3}, 0)$ and verify that its Jacobian at $F(\frac{\pi}{3}, 0)$ equals the one you calculated in part (b). (You might want to look up a few basic trigonometric facts.)

5. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on the interval $(a, b)$, and let $a < c < d < b$.

(a) Suppose that $f'(c) < 0 < f'(d)$. Prove that the restriction of $f$ to $[c, d]$ does not achieve a global minimum at $c$ or at $d$.

(b) Again suppose that $f'(c) < 0 < f'(d)$. Prove that there exists some $p \in (c, d)$ such that $f'(p) = 0$. (In order to receive full credit, please prove any claims you make about the derivative at extremal points.)

(c) Now suppose that $f'(c) < \alpha < f'(d)$. Prove that there exists some $p \in (c, d)$ such that $f'(p) = \alpha$.

6. Let $g : \mathbb{R} \to \mathbb{R}$ be $C^1$. Prove that there exists $\varepsilon > 0$ such that the function $f : [1, 2] \to \mathbb{R}$ given by

$$f(x) = x^3 - x^2 + \varepsilon g(x)$$

is injective. (Hint: You probably want to start by using the Extreme-Value Theorem appropriately.)