1. Consider the following quadratic forms:

\[ f(x, y) = 2x^2 - 4xy + 5y^2, \]
\[ g(x, y) = x^2 + 6xy + y^2, \]
\[ h(x, y) = 16xy. \]

Answer the following questions for each of these forms:

(a) Find a symmetric matrix \( M \) such that the form equals \([x \ y] M [x \ y]\).

(b) Find the eigenvalues of matrix \( M \).

(c) Find an orthonormal basis of eigenvectors.

(d) Find a unitary matrix \( S \) such that \( M = S^{-1}DS \), where \( D \) is a diagonal matrix.

(e) Describe the level sets of the form and state whether the form has a local maximum, local minimum, or neither at \((0, 0)\). (Level sets are solutions to \( f(x, y) = c \) for some \( c \in \mathbb{R} \).)

2. Suppose \( \Psi_1, \Psi_2 : X \to 2^Y \) are compact-valued, upper hemicontinuous correspondences, where \( X \subset \mathbb{R}^n, \ Y \subset \mathbb{R}^m \) for some \( n, m \). Suppose that \( \Psi_1 \cap \Psi_2 \neq \emptyset \) for each \( x \in X \).

(a) Show that \( \Psi_1 \cap \Psi_2 \) is upper hemicontinuous, where \( \Psi_1 \cap \Psi_2 \) is defined by

\[ (\Psi_1 \cap \Psi_2)(x) = \Psi_1(x) \cap \Psi_2(x), \ \forall x \in X \]

(b) Let's now weaken our assumptions a bit: let's assume that \( \Psi_1 \) is only closed-valued, rather then compact-valued. Show that \( \Psi_1 \cap \Psi_2 \) is still upper hemicontinuous.

3. Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be a \( C^1 \) function and define \( F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \) by

\[ F(x, \omega) = f(x_1, x_2) - \left(5\omega_1 + \omega_1^3, 5\omega_2 + \omega_1(1 + 3\omega_1\omega_2)\right). \]

Show that there is a set of Lebesgue measure zero, \( \Omega_0 \subset \mathbb{R}^2 \), such that if \( \omega \not\in \Omega_0 \), then for each \( x_0 \) satisfying \( F(x_0, \omega_0) = 0 \) there is an open set \( U \) containing \( x_0 \), an open set \( V \) containing \( \omega_0 \), and a \( C^1 \) function \( h : V \to U \) such that for all \( \omega \in V \), \( x = h(\omega) \) is the unique element of \( U \) satisfying \( F(x, \omega) = 0 \).
4. The Minimax Theorem is used for proving quite a few important results in economics, for instance, about an outcome of zero-sum games in noncooperative game theory or in analyzing Bayesian estimators in statistical decision theory. Now you have a chance to prove this Minimax Theorem yourself.

Let $X$ and $Y$ be non-empty, closed, bounded and convex subsets of any two Euclidean spaces. Prove that if $f : X \times Y \to \mathbb{R}$ is continuous, and if the sets $\{ z \in X \mid f(z, y) \geq \alpha \}$ and $\{ w \in Y \mid f(x, w) \leq \alpha \}$ are convex for each $(x, y, \alpha) \in X \times Y \times \mathbb{R}$, then

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

(Hint: Start by defining two self-correspondences $\Phi(y) : Y \to 2^Y$ and $\Pi(x) : X \to 2^X$ as

$$\Phi(y) = \text{argmax}_{x \in X} f(x, y)$$

$$\Pi(x) = \text{argmin}_{y \in Y} f(x, y)$$

Then, define self-correspondence $\Psi : X \times Y \to 2^{X \times Y}$ by

$$\Psi(x, y) = \Pi(x) \times \Phi(y).$$

Use Kakutani’s Fixed Point theorem).

5. Show that the closure of a convex set is convex.

6. One of the most useful versions of Separating Hyperplane Theorem is the one on strong separation of convex sets. We say that two sets $A$ and $B$ are strongly separated by a hyperplane if there exists $p \in \mathbb{R}^n$ with $p \neq 0$ such that

$$\sup_{a \in A} p \cdot a < \inf_{b \in B} p \cdot b$$

(In other words, sets are strongly separated if they are contained in the closed halfspaces that are $\epsilon > 0$ away from each other. Notice that another way to show strong separation is to demonstrate existence of two constants $c$ and $d$ together with non-zero vector $p$ such that $p \cdot a \leq c < d \leq p \cdot b$ for all $a \in A$ and for all $b \in B$. Please check Theorem 8 in lecture 13 to make sure you understand how strict separation is different from strong one.)

Of course, strong separation requires a stronger initial assumptions.

(a) Let $A$ and $B$ be non-empty, disjoint, convex subsets of $\mathbb{R}^n$ with $A$ being compact and $B$ closed. Show directly, without invoking Theorem 7 in Lecture 13, that $A$ and $B$ can be strongly separated.\(^1\) (Hint: Look at the set $Y = B - A$. Is it compact? Closed?)

\(^1\)Although you can’t invoke theorem directly, going carefully over its proof will get you far in showing this result.
(b) Demonstrate by means of an example that the requirement that $A$ is compact is essential, it can’t be just closed. In your example, are the sets $A$ and $B$ strictly separated?

7. Consider the following inhomogeneous linear differential equation

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

(a) Write down the corresponding homogeneous equation.

(b) Find the general solution of the homogeneous equation.

(c) Find a particular solution of the original inhomogeneous equation satisfying the initial condition $y(0) = (1, 1)^T$.

(Hint: The integrals can be solved by integrating by parts twice.)

(d) Find the general solution of the original inhomogeneous equation.