Economics 204 Summer/Fall 2010
Final Exam

Answer all of the questions below. Be as complete, correct, and concise as possible. There are 6 questions for a total of 165 points possible; point values for each problem are in parentheses. For questions with subparts, each subpart is worth the same number of points. You have 180 minutes to complete the exam. Use the points as a guide to allocating your time. You may use any result from class with appropriate references unless you are specifically being asked to prove it.

1. (15) Define or state each of the following.
   (a) Brouwer's Fixed Point Theorem
   (b) contraction mapping
   (c) metric

   Solution: See notes.

2. (30) Prove that for every $n \in \mathbf{N}$,

$$\sum_{k=1}^{n} k^3 = \frac{1}{4}\left(n(n+1)\right)^2$$

Solution: For $n = 1$,

$$1 = 1^3 = \frac{1}{4}(1(1+1))^2 = \frac{1}{4}(2^2) = \frac{1}{4}4 = 1$$

For the induction hypothesis, suppose

$$\sum_{k=1}^{n-1} k^3 = \frac{1}{4}\left((n-1)n\right)^2 \quad \text{for some } n-1, n \geq 2$$

Now consider n:

$$\begin{aligned}
\sum_{k=1}^{n} k^3 &= \sum_{k=1}^{n-1} k^3 + n^3 \\
&= \frac{1}{4}\left((n-1)n\right)^2 + n^3 \quad \text{by the induction hypothesis} \\
&= \frac{1}{4}(n^2-n)(n^2-n) + n^3 \\
&= \frac{1}{4}(n^4 - n^3 - n^3 + n^2) + n^3 \\
&= \frac{1}{4}(n^4 + 2n^3 + n^2)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4}(n^2(n^2+2n+1)) \\
&= \frac{1}{4}(n^2(n+1)^2) \\
&= \frac{1}{4}(n(n+1))^2
\end{aligned}$$

By induction, the claim holds for every $n \in \mathbf{N}$.

3. (30) Let X and Y be vector spaces over the same field F and $T \in L(X, Y)$, that is, $T : X \to Y$ is a linear transformation.

   (a) Show that $\ker T$ is a vector subspace of X and that $\operatorname{Im} T$ is a vector subspace of Y.

   Solution: Let $x_1, x_2 \in \ker T$, and $\alpha, \beta \in F$. Then

$$\begin{aligned}
T(\alpha x_1 + \beta x_2) &= \alpha T(x_1) + \beta T(x_2) \quad \text{linearity of } T \\
&= \alpha 0 + \beta 0 \quad \text{since } x_1, x_2 \in \ker T \\
&= 0 + 0 = 0
\end{aligned}$$

   So $\alpha x_1 + \beta x_2 \in \ker T$. Thus $\ker T$ is a vector subspace of X. Similarly, let $y_1, y_2 \in \operatorname{Im} T$ and $\alpha, \beta \in F$. Since $y_1, y_2 \in \operatorname{Im} T$, there exist $x_1, x_2 \in X$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$. Since X is a vector space, $\alpha x_1 + \beta x_2 \in X$. Then

$$\begin{aligned}
T(\alpha x_1 + \beta x_2) &= \alpha T(x_1) + \beta T(x_2) \quad \text{linearity of } T \\
&= \alpha y_1 + \beta y_2
\end{aligned}$$

   which shows that $\alpha y_1 + \beta y_2 \in \operatorname{Im} T$. Thus $\operatorname{Im} T$ is a vector subspace of Y.

   (b) Suppose $\dim X = \dim Y$ and $\ker T = \{0\}$. Show that if $V = \{v_1, \ldots, v_n\}$ is a basis for X, then $\{T(v_1), \ldots, T(v_n)\}$ is a basis for Y.

   Solution: First note that $\dim X = n$, since V is a basis for X and $|V| = n$. Since $\dim Y = \dim X$, $\dim Y = n$ as well. Then it suffices to show that $W = \{T(v_1), \ldots, T(v_n)\}$ is a linearly independent set, since $|W| = n$. To that end, suppose there exist $\alpha_1, \ldots, \alpha_n \in F$ such that

$$\begin{aligned}
0 &= \sum_{i=1}^{n} \alpha_i T(v_i) \\
\Rightarrow 0 &= \sum_{i=1}^{n} \alpha_i T(v_i) = T(\sum_{i=1}^{n} \alpha_i v_i) \quad \text{linearity of } T
\end{aligned}$$

   Thus $\sum_{i=1}^{n} \alpha_i v_i \in \ker T = \{0\}$, so $\sum_{i=1}^{n} \alpha_i v_i = 0$. Since $\{v_1, \ldots, v_n\}$ are linearly independent, $\alpha_i = 0$ for each i. Thus $\{T(v_1), \ldots, T(v_n)\}$ are linearly independent.

4. (30) Consider $\mathbf{R}$ with the usual metric.

(a) Let $C = \{\frac{n}{n^2+1} : n = 0, 1, 2, 3, \ldots\}$. Show directly from the definition that C is compact.

(**Note:** An otherwise correct answer that does not use the open cover definition will receive 10 points.)

Solution: Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover of C. Then there exists $\lambda_0 \in \Lambda$ such that $0 = \frac{0}{0^2+1} \in U_{\lambda_0}$. Since U_{λ_0} is an open set containing 0, there exists $\varepsilon > 0$ such that $B_\varepsilon(0) \subseteq U_{\lambda_0}$. Notice that the sequence $\{\frac{n}{n^2+1}, n = 1, 2, 3, \ldots\}$ converges to 0 as $n \to \infty$, so there exists N such that for all $n > N$, $\frac{n}{n^2+1} \in B_\varepsilon(0) \subseteq U_{\lambda_0}$. By definition of open cover, there exist $\lambda_1, \ldots, \lambda_N \in \Lambda$ such that $\frac{n}{n^2+1} \in U_{\lambda_n}$ for $n = 1, \ldots, N$. Thus

$$C \subseteq U_{\lambda_0} \cup U_{\lambda_1} \cup \cdots \cup U_{\lambda_N}$$

So $\{U_{\lambda_0}, U_{\lambda_1}, \ldots, U_{\lambda_N}\}$ is a finite subcover of $\{U_\lambda : \lambda \in \Lambda\}$. Since the original open cover was arbitrary, C is compact.

Here is an argument for 10 points. Notice that this argument is actually harder. Since $C \subseteq [0, 1]$ and $[0, 1]$ is compact, it suffices to show that C is closed. Let $\{x_n\}$ be a sequence of elements of C, and suppose $x_n \to x \in [0, 1]$. Since $x_n \in C$ for each n, either $\{x_n\}$ is eventually constant or $x_n \to 0$. If $\{x_n\}$ is eventually constant, then $x = \frac{k}{k^2+1}$ for some $k = 0, 1, 2, \ldots$, so $x \in C$. If $x = 0$, then $x \in C$. Thus C is closed.

(b) Let $C_1 = C \setminus \{0\} = \{\frac{n}{n^2+1} : n = 1, 2, 3, \ldots\}$. Is C_1 compact? Justify your answer.

(**Note:** Answers with no justification will receive no points.)

Solution: No. Notice that the sequence $\{x_n = \frac{n}{n^2+1}, n = 1, 2, 3, \ldots\}$ converges to 0 as $n \to \infty$, so any subsequence of $\{x_n\}$ converges to 0 as well. But since $0 \notin C_0$ and $\{x_n\} \subseteq C_0$, $\{x_n\}$ is a sequence in C_0 with no subsequence that converges to an element of C_0. A set in $\mathbf{R}$ is compact if and only if it is sequentially compact, thus C_0 is not compact.

Alternatively, note that $x_n \to 0$, $\{x_n\} \subseteq C_0$ but $0 \notin C_0$ implies C_0 is not closed. Every compact subset of $\mathbf{R}$ must be closed, hence C_0 is not compact.

5. (30) Let $f : A \to \mathbf{R}^m$ be continuous, where $A \subseteq \mathbf{R}^n$ is open and convex. Show that if f is differentiable on A and $\|df_x\|$ is bounded on A, then f is uniformly continuous on A.

Solution: Let $M > 0$ be a bound on $\|df_x\|$ for $x \in A$, that is, $\|df_x\| \leq M$ for every $x \in A$. Fix $\varepsilon > 0$. Set $\delta = \varepsilon/M$, and let $x, y \in A$ with $\|x - y\| < \delta$. Since A is convex and $x, y \in A$, $\ell(x, y) = \{z : z = \alpha x + (1 - \alpha)y, \alpha \in [0, 1]\} \subseteq A$. By the Mean Value Theorem, there exists $z \in \ell(x, y)$ such that

$$\|f(x) - f(y)\| \leq \|df_z\| \|x - y\|$$

Then

$$\begin{aligned}\|f(x)-f(y)\| &\leq \|df_z\|\|x-y\| \\ &\leq M\|x-y\| \\ &< M\delta \\ &= \varepsilon\end{aligned}$$

Hence f is uniformly continuous on A.

6. (30) Suppose $\Psi_1, \Psi_2 : X \to 2^Y$ are closed-valued, upper hemicontinuous correspondences, where $X \subseteq \mathbf{R}^n, Y \subseteq \mathbf{R}^m$ for some n, m. Suppose that $\Psi_1(x) \cap \Psi_2(x) \neq \emptyset$ for each $x \in X$. Show that $\Psi_1 \cap \Psi_2$ is upper hemicontinuous, where $\Psi_1 \cap \Psi_2 : X \to 2^Y$ is defined by

$$(\Psi_1 \cap \Psi_2)(x) = \Psi_1(x) \cap \Psi_2(x) \quad \forall x \in X$$

(**Note:** For full credit, the answer will have to directly use the definition of upper hemicontinuity. An otherwise correct answer that uses alternative characterizations of upper hemicontinuity will receive 50% credit, provided any necessary additional assumptions are clearly stated.)

Solution: Fix $x \in X$ and let $O \subseteq \mathbf{R}^m$ be an open set such that

$$\Psi_1(x) \cap \Psi_2(x) \subseteq O$$

Since $\Psi_1(x), \Psi_2(x)$ are closed sets, $\Psi_2(x) \setminus O$ is closed, and

$$\Psi_1(x) \cap (\Psi_2(x) \setminus O) = \emptyset$$

Then there exist open sets $V_1, V_2 \subseteq \mathbf{R}^m$ such that $V_1 \cap V_2 = \emptyset$ with

$$\Psi_1(x) \subseteq V_1, \quad \Psi_2(x) \setminus O \subseteq V_2$$

Thus $\Psi_2(x) \subseteq O \cup V_2$, and $O \cup V_2$ is an open set. Since Ψ_1 and Ψ_2 are upper hemicontinuous, there exist open sets $U_1, U_2 \subseteq X$ with $x \in U_1, x \in U_2$ such that

$$\Psi_1(x') \subseteq V_1 \;\; \forall x' \in U_1 \text{ and } \Psi_2(x') \subseteq O \cup V_2 \;\; \forall x' \in U_2$$

Notice that $x \in U_1 \cap U_2$, so $U_1 \cap U_2 \neq \emptyset$, and $U_1 \cap U_2$ is open. Let $\hat{x} \in U_1 \cap U_2$. Then $\Psi_1(\hat{x}) \subseteq V_1$, $\Psi_2(\hat{x}) \subseteq O \cup V_2$, and $\Psi_1(\hat{x}) \cap \Psi_2(\hat{x}) \neq \emptyset$. Since $V_1 \cap V_2 = \emptyset$, this implies

$$\Psi_1(\hat{x}) \cap \Psi_2(\hat{x}) \subseteq O$$

Thus $\Psi_1 \cap \Psi_2$ is upper hemicontinuous.

Here is an argument for half credit using the sequential characterization of uhc. Suppose Ψ_1 is compact-valued. Then since Ψ_2 is closed-valued, $\Psi_1 \cap \Psi_2$ is compact-valued. Now suppose $x_n \to x$ and $y_n \in (\Psi_1 \cap \Psi_2)(x_n)$ for each n. Since Ψ_1 is upper hemicontinuous and compact-valued, there is a subsequence $\{y_{n_k}\}$ such that $y_{n_k} \to y$ with $y \in \Psi_1(x)$. Ψ_2 is closed-valued and upper hemicontinuous, so has closed graph. Then $(x_{n_k}, y_{n_k}) \in$ graph Ψ_2 and $(x_{n_k}, y_{n_k}) \to (x, y)$, so $(x, y) \in$ graph Ψ_2, that is, $y \in \Psi_2(x)$. So $y \in \Psi_1(x) \cap \Psi_2(x) = (\Psi_1 \cap \Psi_2)(x)$. Hence $\Psi_1 \cap \Psi_2$ is upper hemicontinuous.