Econ 204
Set Formation and the Axiom of Choice

In this supplement, we discuss the rules underlying set formation and the Axiom of Choice.

We generally begin with a set of elements, such as the natural numbers $\mathbf{N}$, the rational numbers $\mathbf{Q}$, the real numbers $\mathbf{R}$, or an abstract set like the set X of all points of an unspecified metric space.

Given any set X, we can form 2^X, often called the *power set* of X; 2^X is the set of all subsets of X. Thus, we can form the set $\mathbf{N}$ of all natural numbers, $2^{\mathbf{N}}$, the set of all subsets of $\mathbf{N}$; $\emptyset$, $\{1,2\}$, $\{2,4,6,\ldots\}$ are elements of $2^{\mathbf{N}}$.

We can also form $2^{2^{\mathbf{N}}} = 2^{\left(2^{\mathbf{N}}\right)}$, the set of all subsets of the set of all subsets of the natural numbers. An element of $2^{2^{\mathbf{N}}}$ is a set of subsets of the natural numbers; for example, $\{\emptyset\}$, $\{\emptyset, \mathbf{N}\}$, $\{\{1\},\{2\},\{2,4,6,\ldots\}\}$ and $\{\{2\},\{4\},\{6\},\ldots\}$ are elements of $2^{2^{\mathbf{N}}}$.

Let X be any set, and $P(x)$ a mathematical statement about a variable x. Then

$$\{x \in X : P(x)\}$$

is a set; it is the collection of all elements x of X such that the statement $P(x)$ is true. For example, if f is a function from $[a,b]$ to $\mathbf{R}$, then $\{t \in [a,b] : f(t) < 7\}$ is a valid set; it consists of all those elements t in the interval $[a,b]$ such that $f(t) < 7$. The statement P can be complex. In particular, it can include quantifiers. For example,

$$\{x \in [0,1] : \forall y \in [0,1]\ x \geq y\}$$

is a valid set; it equals $\{1\}$.

$$\{x \in (0,1) : \forall y \in (0,1)\ x > y\}$$

is also a valid set; it equals the empty set. The set of all upper bounds for $X \subseteq 2^{\mathbf{R}}$ is

$$U = \{u \in \mathbf{R} : u \geq x \quad \forall x \in X\}$$

In order to avoid Russell's Paradox[1], one needs to exercise a little care in forming sets. In practice, the things that a working economist needs to

[1]In the early days of set theory, mathematicians were somewhat cavalier about what

do are always legal. You can always apply the power set construction an arbitrary finite number of times, and use quantifiers of the form $\forall x \in X$ as long as X is a set formed by taking at most a finite number of applications of the power set operation. Thus,

$$2^{\left(2^{\left(2^{\mathbf{R}}\right)}\right)}$$

is fine. Working economists have no interest in sets[2] like

$$Y = \{1, \{1\}, \{\{1\}\}, \{\{\{1\}\}\}, \ldots\}$$

which involve unbounded applications of the power set construction.

A function $f : X \to Y$ is defined in terms of its graph

$$G_f = \{(x, y) : y = f(x)\} \subseteq X \times Y = \{(x, y) : x \in X, y \in Y\}$$

so $G_f \in 2^{X \times Y}$. The fact that f is a function says that

$$((x, y) \in G_f \wedge (x, z) \in G_f) \Rightarrow (y = z)$$

The collection of all functions mapping X to Y is thus a subset of $2^{X \times Y}$, and is thus an element of $2^{2^{X \times Y}}$. Therefore, we can write quantifiers over functions. For example,

$$\forall f : \mathbf{N} \to \mathbf{R}\ \exists x \in \mathbf{R} \text{ s.t. } \nexists n \in \mathbf{N} \text{ s.t. } f(n) = x$$

states that there is no function mapping $\mathbf{N}$ onto $\mathbf{R}$.

Suppose we are given a set Λ and a function $G : \Lambda \to 2^X$ for some set X. Then the Axiom of Choice asserts

$$(G(\lambda) \neq \emptyset\ \forall \lambda \in \Lambda) \Rightarrow (\exists f : \Lambda \to X \text{ s.t. } f(\lambda) \in G(\lambda)\ \forall \lambda \in \Lambda)$$

In other words, if I can chose an element of $G(\lambda)$ one λ at a time, I can choose a *function* $f : \Lambda \to X$ such that $f(\lambda) \in G(\lambda)$ for all $\lambda \in \Lambda$. For example, suppose that for all $n \in \mathbf{N}$, $B_{1/n}(y) \cap X \neq \emptyset$. Then the Axiom of Choice tells us that there is a sequence (recall a sequence is a function whose domain is $\mathbf{N}$) $\{x_n\}$ of elements of x such that $x_n \in B_{1/n}(y)$ (and hence $x_n \to y$).

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constituted a set. Bertrand Russell point out that if the collection of all sets is a set Ω, then one can form $E = \{X \in \Omega : X \notin X\}$, the set of all sets which are not elements of themselves. Is $E \in E$? If so, then $E \notin E$, contradiction; if not, then $E \in E$, again a contradiction. Thus, one needs to define the notion of "set" in such a way that the collection of all sets is not a "set."

[2] Y is a valid set, but one needs to exercise caution with respect to quantifiers of the form $\forall y \in Y$.