Econ 204
Corrections to de la Fuente

1. On page 23, de la Fuente presents two definitions of correspondence. In the second definition, de la Fuente requires that for all $x \in X$, $\Psi(x) \neq \emptyset$. The first definition simply says that Ψ is a function from X to 2^Y, the collection of all subsets of Y, and appears to suggest that this implies that $\Psi(x) \neq \emptyset$; since $\emptyset \in 2^Y$, this is not correct. The first definition should have said "a correspondence is a function from X to 2^Y such that for all $x \in X$, $\Psi(x) \neq \emptyset$."

2. Theorem 5.2, page 64, should read as follows:

   Theorem 1 (5.2') *Let (X, d) and (Y, ρ) be two metric spaces, $A \subseteq X$, $f : A \to Y$, and x^0 a limit point of A. Then f has limit y^0 as $x \to x^0$ if and only if for every sequence $\{x_n\}$ that converges to x^0 in (X, d) with $x_n \in A$ for every n and $x_n \neq x^0$, the sequence $\{f(x_n)\}$ converges to y^0 in (Y, ρ).*

   Comment: As stated in de la Fuente, the metric space (X, d) is the ambient space, so every limit point of X must be an element of X; there is nothing outside of X to which a sequence in X can converge. Thus, as stated in de la Fuente, we must have $x^0 \in X$ and thus, x^0 must be in the domain of f. The revised statement just given allows x^0 to lie outside the domain of f.

3. De la Fuente uses a weaker definition of homeomorphism (Definition 6.20, page 74) than most texts; usually, a homeomorphism is required to be a surjection. For example, the injection map $I : [0, 1] \to \mathbf{R}$ defined by $I(x) = x$ would not be called a homeomorphism in most texts because it is not onto, but it is a homeomorphism under de la Fuente's weaker definition. This creates trouble in Theorem 6.21(ii). $[0, 1]$ is an open set in the metric space $[0, 1]$, but its image $I([0, 1]) = [0, 1]$ is *not* an open set in $\mathbf{R}$, so Theorem 6.21 is false as stated. Theorem 6.21 is true if we assume that $f : (X, d) \to (Y, \rho)$ is one-to-one *and onto*, or if we replace the phrase "its image $f(A_X)$ is open in (Y, ρ)" in part (ii) with "its image $f(A_X)$ is open in $\left(f(X), \rho|_{f(X)}\right)$."

4. The proof of Theorem 7.12 is a bit disorganized and hard to follow. In the second bullet on page 84, de la Fuente assumes that $\{f_n\}$ converges to f in the sup norm, but this is not proven until the third bullet. Since the third bullet does not use the continuity of the limit function f, we could simply switch the second and third bullets to get a correct (but awkward) proof. Here is a better alternative to the second and third bullets:

    - Fix $\varepsilon > 0$. Since the sequence $\{f_n\}$ is Cauchy in the sup norm, there exists N such $n, m > N \Rightarrow \|f_n - f_m\|_s < \varepsilon/3$. Fix $m > N$. Then $n > N \Rightarrow \|f_n - f_m\|_s < \varepsilon/3$, so for each $x \in X$, $|f(x) - f_m(x)| = \lim_{n\to\infty} |f_n(x) - f_m(x)| \leq \varepsilon/3$. Since x is arbitrary, $m > N \Rightarrow \|f - f_m\|_s \leq \varepsilon/3 < \varepsilon$, so $\|f - f_m\|_s \to 0$, i.e. $\{f_m\}$ converges to f in the sup norm.
    - To see that f is continuous, fix $x_0 \in X$; let N be as in the previous bullet and take $m = N + 1$. Since f_{N+1} is continuous, there exists $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f_{N+1}(x) - f_{N+1}(x_0)| < \varepsilon/3$. Therefore, if $|x - x_0| < \delta$,

    $$\begin{aligned} &|f(x) - f(x_0)| \\ &\quad \leq \quad |f(x) - f_{N+1}(x)| + |f_{N+1}(x) - f_{N+1}(x_0)| + |f_{N+1}(x_0) - f(x_0)| \\ &\quad < \quad \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &\quad = \quad \varepsilon \end{aligned}$$

      Therefore, f is continuous, which completes the proof.

5. Some of the wording on page 85 is imprecise. Just before the formal definition of a contraction, de la Fuente says "We say that an operator is a contraction if its application to any two points brings them closer to each other;" this suggests that the definition should be either "for all $x, y \in X$ with $x \neq y$, $d(T(x), T(y)) < d(x, y)$" or "for all $x, y \in X$, $d(T(x), T(y)) \leq d(x, y)$." De la Fuente's formal definition of a contraction is correct, and stronger than the informal statement. Here is an equivalent rephrasing of the formal definition of a contraction:

   Definition 2 Let (X, d) be a metric space and $T : X \to X$. We say that T is a *contraction* if there exists $\beta < 1$ such that for all $x, y \in X$, $d(T(x), T(y)) \leq \beta d(x, y)$.

In the definition, it is very important that there exist a single $\beta < 1$ that works for all $x, y \in X$.

Example 7.15 is imprecisely stated. Replace the first sentence by "Let $f : [a, b] \to [a, b]$ be a continuously differentiable function such that $0 \leq f'(x) < 1$ for all $x \in [a, b]$." With that change, f is indeed a contraction, although the argument uses results that have not been covered at this point in the book. Since $f'(x)$ is a continuous function on a closed bounded interval, it attains its maximum at some point x_0. Let $\beta = f'(x_0)$; $\beta < 1$. Given any $x \neq y \in [a, b]$, by the Mean Value Theorem, there exists $c \in (x, y)$ such that $\frac{f(y)-f(x)}{y-x} = f'(c) \in [0, \beta]$, so $|f(y) - f(x)| \leq \beta|y - x|$, so f is a contraction.

In Figure 2.11 on page 86, the function that is graphed is not a contraction: as pictured, its slope is greater than one near the point a.

6. In Theorem 7.16 (page 86), you need to add the assumption that $X \neq \emptyset$.[1] Alternatively, de la Fuente could have required $X \neq \emptyset$ as part of the definition of a metric space.

7. I'm not sure whether Theorem 7.18 (page 88) is correct as stated, but the proof given proves only the following weaker theorem:

Theorem 3 (7.18') *Let (X, d) and (Ω, ρ) be two metric spaces, and $T(x, \alpha)$ a function $X \times \Omega \to X$. If (X, d) is complete, if T is continuous in α, and there is $\beta < 1$ such that, for each $\alpha \in \Omega$, the function T_α defined by $T_\alpha(x) = T(x, \alpha)$ for each $x \in X$ is a contraction with modulus β, then the solution function $z : \Omega \to X$, with $x^* = z(\alpha)$, which gives the fixed point as a function of the parameters, is continuous.*

To understand the difference, note that de la Fuente's statement requires that T_α be a contraction, but implicitly allows β_α, the modulus of the contraction T_α, to vary with α; we might have $\beta_\alpha < 1$ for each α, but $\sup\{\beta_\alpha : \alpha \in \Omega\} = 1$. The revised statement and de la Fuente's proof require that there be a single $\beta \in (0, 1)$ such that T_α is a contraction with modulus β for all α.

[1] There is exactly one function from $\emptyset$ to itself, it is a contraction, but it has no fixed point.

8. De la Fuente's definition of totally bounded (page 92) is not the usual definition. Here are de la Fuente's definition (in a slightly condensed form) and the standard definition:

   (a) *Definition* (de la Fuente): $A \subset (X, d)$ is totally bounded if, for every $\varepsilon > 0$, there exists $x_1, \ldots, x_n \in X$ such that

   $$A \subset B(x_1, \varepsilon) \cup \cdots \cup B(x_n, \varepsilon)$$

   (b) *Definition* (standard): $A \subset (X, d)$ is totally bounded if, for every $\varepsilon > 0$, there exists $x_1, \ldots, x_n \in A$ such that

   $$A \subset B(x_1, \varepsilon) \cup \cdots \cup B(x_n, \varepsilon)$$

   The only difference between the two is that de la Fuente allows the points $x_1, \ldots, x_n$ to come from the ambient metric space X, while the standard definition requires that they come from the set in question, A. This gets him into trouble in the proof of Theorem 8.11. With his definition of totally bounded, the points $x_1, \ldots, x_n$ lie in X but not necessarily in A; since the open cover is a cover of A but not necessarily of X, it does not follow that $B_\varepsilon(x_i) \subseteq U_i$ as claimed. Another reason to prefer the standard definition is that it is intrinsic to A. A set A can be a subspace of many metric spaces; under the standard definition, whether or not a set A is totally bounded depends only on the set A and the metric restricted to A. In de la Fuente's definition, it appears that a set A might be totally bounded when viewed as a subspace of a metric space (X, d), but not as a subspace of (X', d'), even if d and d' induce the same metric on A. In fact, the two definitions are equivalent:

   Proposition 4 *Suppose A is a set in a metric space (X, d). Then A is totally bounded in de la Fuente's sense if and only if it is totally bounded in the standard sense.*

   Proof: If A is totally bounded in the standard sense, it is trivially totally bounded in de la Fuente's sense. To prove the converse, suppose that A is totally bounded in de la Fuente's sense. Fix $\varepsilon > 0$. There exist $x_1, \ldots, x_n \in X$ such that

   $$A \subseteq \cup_{i=1}^n B_{\varepsilon/2}(x_i)$$

Without loss of generality, we may assume that for each i, $A \cap B_{\varepsilon/2}(x_i) \neq \emptyset$. For each i, choose $a_i \in A \cap B_{\varepsilon/2}(x_i)$. Then

$$\begin{aligned} a \in A &\Rightarrow a \in B_{\varepsilon/2}(x_i) \text{ for some } i \in \{1, \ldots, n\} \\ &\Rightarrow d(a, x_i) < \frac{\varepsilon}{2}, \;\; d(a_i, x_i) < \frac{\varepsilon}{2} \\ &\Rightarrow d(a, a_i) < \varepsilon \end{aligned}$$

Therefore,

$$A \subseteq \cup_{i=1}^{n} B_\varepsilon(a_i)$$

so A is totally bounded in the standard sense. ■

9. De la Fuente's definition of arcwise-connectedness (Definition 9.4 on page 103) is stronger than the usual one. The usual definition is as follows:

   Definition 5 (9.4') A set B in a metric space (X, d) is said to be arcwise-connected if, for every pair of points $x, y \in B$, there is an interval $[a, b] \subset \mathbf{R}$ and a continuous function $f : [a, b] \to B$ such that $f(a) = x$ and $f(b) = y$.

10. Theorem 10.8 (all norms on $\mathbf{R}^n$ are Lipschitz equivalent) is correct, but the proof is incorrect. Problem 6.8 showed that if $(X, \|\cdot\|)$ is a normed space, then $\|\cdot\|$ is a continuous function on $(X, \|\cdot\|)$; note that this is true for *any* normed space, including infinite-dimensional normed spaces, which support inequivalent norms. $C = \{x \in X : \|x\|_E\}$ is compact in the topology generated by $\|x\|_E$, but it does *not* follow that $\|x\|$ achieves its min or max on C, since we don't know that $\|\cdot\|$ is continuous *with respect to* $\|\cdot\|_E$. Indeed, continuity of $\|\cdot\|$ with respect to $\|\cdot\|_E$ is equivalent to the statement that there exists M such that $\|x\| \leq M\|x\|_E$, which is part of what we are trying to prove. Nor does it follow that C is compact in the topology generated by $\|\cdot\|$. The correct proof makes greater use of the finite-dimensionality of $\mathbf{R}^n$.

11. In the statement of Theorem 1.2 on page 118, "coefficients not all zero" should read "all coefficients nonzero;" notice that this is what the proof

proves.[2]

12. In the statement of Theorem 11.11 on page 113, $\sum_{i=1}^n \Psi_i(x)$ has not been defined. Given sets $A_1, \ldots, A_n$,

$$\sum_{i=1}^{n} A_i = \{a_1 + \cdots + a_n : a_1 \in A_1, \ldots, a_n \in A_n\}$$

is the collection of all sums of elements, with one element taken from each set. Thus,

$$\sum_{i=1}^{n} \Psi(x) = \{a_1 + \cdots + a_n : a_1 \in \Psi_1(x), \ldots, a_n \in \Psi_n(x)\}$$

defines a correspondence.

13. On page 119, de la Fuente says every nontrivial vector space ($V \neq \{0\}$) has a Hamel basis. This is correct, but the trivial vector space $V = \{0\}$ also has a Hamel basis; indeed, the empty set $\emptyset$ is a Hamel basis for $V = \{0\}$. In the same sentence of de la Fuente, "have the same cardinal number," which was not previously defined, means "are numerically equivalent."

14. On page 146, the notation $|A - \lambda I|$ means the determinant of the matrix $A - \lambda I$; I don't think determinants are defined in the book.

15. The discussion at the bottom of page 150 and the top of page 151 is garbled. In the fourth line of page 151, de La Fuente establishes that $M_a Py = \lambda Py$ and $M_a x = \lambda x$, and concludes that $x = Py$. To see this need not be true, suppose that M_a is the 2×2 identity matrix. Then take

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad Py = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Note that $M_a x = 1 \cdot x$ and $M_a Py = 1 \cdot Py$, but it is definitely not the case that $x = Py$; indeed, x is not even a scalar multiple of Py. In fact,

[2] Also, to be really fussy, the word "nonzero" in the second sentence can be dropped. The zero vector has a unique representation as a linear combination with all coefficients nonzero of a finite subset of $\mathbf{b}$; $\emptyset$ is a finite subset of $\mathbf{b}$, $\sum_{v \in \emptyset} 1 \cdot v = 0$, and since there are no coefficients, all of them are nonzero.

the claim in the second last line of page 150, that the eigenvectors of M_a and M_b represent the same elements of V in the two bases, is not correct. The following is correct:

Proposition 6 *Suppose that V is an n-dimensional vector space, and $T : V \to V$ is a linear map, with matrices M_A and M_B with respect to two bases $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$.*

(a) λ is an eigenvalue of M_A if and only if it is an eigenvalue of M_B.

(b) If M_A has n distinct eigenvalues, the eigenvectors of M_A represent the same elements of V, up to scalar multiplication, as do the eigenvectors of M_B.

(c) If $\lambda \in \mathbf{C}$, let $X_{A\lambda} = \{x \in \mathbf{R}^n : M_A x = \lambda x\}$ and $X_{B\lambda} = \{x \in \mathbf{R}^n : M_B x = \lambda x\}$. Then

$$\begin{aligned}&\{x_1 a_1 + \cdots + x_n a_n : x \in X_{A\lambda}\}\\ &\quad = \{x_1 b_1 + \cdots + x_n b_n : x \in X_{B\lambda}\}\\ &\quad = \{v \in V : Tv = \lambda v\}\end{aligned}$$

The proof follows from the arguments given in de la Fuente.

16. Here is a list of typos and other minor errors:

(a) Page 62, proof of Theorem 4.11: $\Rightarrow$ and $\Leftarrow$ switched.

(b) Page 78, statement of Theorem 6.27: (2) should read $f(x^+) \leq f(y^-)$.

(c) Page 84, proof of Theorem 7.12: The second section of the proof, establishing continuity of f, states "Because $\{f_n\} \to f$ in the sup norm" but this is not established until later in the third section of the proof.

(d) Page 104, proof of Theorem 9.5: "We will now show that D is not connected" at the end of the first paragraph should read "We will now show that C is not connected".

(e) Page 111, proof of Theorem 11.3: Near end of first section, phrase "for each n with $n_k < n_{k+1}$" should read "for each n with $n > n_k$." The following sentence is a little unclear, and would be better rephrased as follows: "Given $\varepsilon > 0$, find k such that $\frac{1}{k} < \varepsilon$; for $n > n_k$, $|y_n - y| < \frac{1}{k} < \varepsilon$, so $\{y_n\}$ converges to y."

(f) Page 122, text following Problem 2.2: misprint in equation for $T(x)$. Last $=$ should be a $+$.

(g) Page 123, last line, $\alpha y_1 + \beta x_2$ should read $\alpha y_1 + \beta y_2$.

(h) Page 129, proof of Theorem 3.3: Last equation $T(x') = T(x')$ should read $T(x') = T(x'')$.

(i) Page 150, discussion at bottom of page. Last item in last equation, $\mid M_a \lambda I \mid$ should read $\mid M_a - \lambda I \mid$.

(j) Page 676, solution to Problem 8.18: $A_n \subseteq A_{n+1}$ in first paragraph should read $A_{n+1} \subseteq A_n$