Econ 204
Differential Equations

In this supplement, we use the methods we have developed so far to study differential equations.

1 Existence and Uniqueness of Solutions

Definition 1 A *differential equation* is an equation of the form $y'(t) = F(y(t), t)$ where U is an open subset of $\mathbf{R}^n \times \mathbf{R}$ and $F : U \to \mathbf{R}^n$. An *initial value problem* is a differential equation combined with an initial condition $y(t_0) = y_0$ with $(y_0, t_0) \in U$. A *solution* of the initial value problem is a differentiable function $y : (a, b) \to \mathbf{R}^n$ such that $t_0 \in (a, b)$, $y(t_0) = y_0$ and, for all $t \in (a, b)$, $\frac{dy}{dt} = F(y(t), t)$. The *general solution* of the differential equation is the family of all solutions for all initial values $(y_0, t_0) \in U$.

Theorem 2 *Consider the initial value problem*

$$y'(t) = F(y(t), t),\ y(t_0) = y_0 \tag{1}$$

Let U be an open set in $\mathbf{R}^n \times \mathbf{R}$ containing (y_0, t_0).

- *Suppose $F : U \to \mathbf{R}^n$ is continuous. Then the initial value problem has a solution.*
- *If, in addition, F is Lipschitz in y on U (i.e. there is a constant K such that for all $(y, t), (\hat{y}, t) \in U$, $|F(y, t) - F(\hat{y}, t)| \leq K|y - \hat{y}|$), then there is an interval (a, b) containing t_0 such that the solution is unique on (a, b).*

Proof: We will limit our proof to the case in which F is Lipschitz; for the general case, see Coddington and Levinson [1]. Since U is open, we may choose $r > 0$ such that

$$R = \{(y, t) : |y - y_0| \leq r, |t - t_0| \leq r\} \subseteq U$$

Given the Lipschitz condition, we may assume that $|F(y,t) - F(\hat{y},t)| \leq K|y - \hat{y}|$ for all $(y,t), (\hat{y},t) \in R$. Let

$$\delta = \min\left\{\frac{1}{2K}, \frac{r}{M}\right\}$$

We claim the initial value problem has a unique solution on $(t_0 - \delta, t_0 + \delta)$.

Let C be the space of continuous functions from $[t_0 - \delta, t_0 + \delta]$ to $\mathbf{R}^n$, endowed with the sup norm

$$\|f\|_\infty = \sup\{|f(t)| : t \in [t_0 - \delta, t_0 + \delta]\}$$

Let

$$S = \{z \in C([t_0 - \delta, t_0 + \delta]) : (z(s), s) \in R \text{ for all } s \in [t_0 - \delta, t_0 + \delta]\}$$

S is a closed subset of the complete metric space C, so S is a complete metric space. Consider the function $I : S \to C$ defined by

$$I(z)(t) = y_0 + \int_{t_0}^t F(z(s), s)\, ds$$

$I(z)$ is defined and continuous because F is bounded and continuous on R. Observe that if $(z(s), s) \in R$ for all $s \in [t_0 - \delta, t_0 + \delta]$, then

$$\begin{aligned} |I(z)(t) - y_0| &= \left|\int_{t_0}^t F(z(s), s)\, ds\right| \\ &\leq |t - t_0| \max\{|F(y,s)| : (y,s) \in R\} \\ &\leq \delta M \\ &\leq r \end{aligned}$$

so $(I(z)(t), t) \in R$ for all $t \in [t_0 - \delta, t_0 + \delta]$. Thus, $I : S \to S$.

Given two functions $x, z \in S$ and $t \in [t_0 - \delta, t_0 + \delta]$,

$$\begin{aligned} |I(z)(t) - I(x)(t)| &= \left|y_0 + \int_{t_0}^t F(z(s), s)\, ds - y_0 - \int_{t_0}^t F(x(s), s)\, ds\right| \\ &= \left|\int_{t_0}^t \left(F(z(s), s) - F(x(s), s)\right)\, ds\right| \\ &\leq |t - t_0| \sup\{|F(z(s), s) - F(x(s), s)| : s \in [t_0 - \delta, t_0 + \delta]\} \\ &\leq \delta K \sup\{|z(s) - x(s)| : s \in [t_0 - \delta, t_0 + \delta]\} \\ &\leq \frac{\|z - x\|_\infty}{2} \end{aligned}$$

Therefore, $\|I(z) - I(x)\|_\infty \leq \frac{\|z-x\|_\infty}{2}$, so I is a contraction. Since S is a complete metric space, I has a unique fixed point $y \in S$. Therefore, for all $t \in [t_0 - \delta, t_0 + \delta]$, we have

$$y(t) = \int_{t_0}^{t} F(y(s), s)\, ds$$

F is continuous, so the Fundamental Theorem of Calculus implies that

$$y'(t) = F(y(t), t)$$

for all $t \in (t_0 - \delta, t_0 + \delta)$. Since we also have

$$y(t_0) = y_0 + \int_{t_0}^{t_0} F(y(s), s)\, ds = y_0$$

y (restricted to $(t_0 - \delta, t_0 + \delta)$) is a solution of the initial value problem.

On the other hand, suppose that $\hat{y}$ is any solution of the initial value problem on $(t_0 - \delta, t_0 + \delta)$. It is easy to check that $(\hat{y}(s), s) \in R$ for all $s \in (t_0 - \delta, t_0 + \delta)$, so we have $|F(\hat{y}(s), s)| \leq M$; this implies that $\hat{y}$ has a extension to a continuous function (still denoted $\hat{y}$) in S. Since $\hat{y}$ is a solution of the initial value problem, the Fundamental Theorem of Calculus implies that $I(\hat{y}) = \hat{y}$. Since y is the unique fixed point of I, $\hat{y} = y$. ■

Example 3 Consider the initial value problem

$$y'(t) = 1 + y^2(t),\ \ y(0) = 0$$

Here, we have $F(y, t) = 1 + y^2$ which is Lipschitz in y over sets $U = V \times \mathbf{R}$, provided that V is bounded, but not over all of $\mathbf{R} \times \mathbf{R}$. The theorem tells us that the initial value problem has a unique solution over some interval of times (a, b), with $0 \in (a, b)$. We claim the unique solution is $y(t) = \tan t$. To see this, note that

$$\begin{aligned} y'(t) &= \frac{d}{dt} \tan t \\ &= \frac{d}{dt} \frac{\sin t}{\cos t} \\ &= \frac{\cos t \cos t - \sin t(-\sin t)}{\cos^2 t} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\cos^2 t + \sin^2 t}{\cos^2 t} \\
&= 1 + \frac{\sin^2 t}{\cos^2 t} \\
&= 1 + \tan^2 t \\
&= 1 + (y(t))^2 \\
y(0) &= \tan 0 \\
&= 0
\end{aligned}
$$

Notice that $y(t)$ is defined for $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, but

$$\lim_{t \to -\frac{\pi}{2}^+} y(t) = -\infty \text{ and } \lim_{t \to \frac{\pi}{2}^-} y(t) = \infty$$

Thus, the solution of the initial value problem cannot be extended beyond the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, because the solution "blows up" at $-\pi/2$ and $\pi/2$.

Example 4 Consider the initial value problem

$$y'(t) = 2\sqrt{|y|}, \; y(0) = 0 \tag{2}$$

The function $F(y,t) = 2\sqrt{|y|}$ is not locally Lipschitz in y at $y = 0$:

$$
\begin{aligned}
2\sqrt{|y|} - 2\sqrt{0} &= 2\sqrt{|y|} \\
&= \frac{2}{\sqrt{|y|}}|y| \\
&= \frac{2}{\sqrt{|y|}}|y - 0|
\end{aligned}
$$

which is not a bounded multiple of $|y - 0|$. Given any $\alpha \geq 0$, let

$$y_\alpha(t) = \begin{cases} 0 & \text{if } \; t \leq \alpha \\ (t-\alpha)^2 & \text{if } \; t \geq \alpha \end{cases}$$

We claim that y_α is a solution of the initial value problem (2) for every $\alpha \geq 0$. For $t < \alpha$, $y'_\alpha(t) = 0 = \sqrt{|0|} = 2\sqrt{|y_\alpha(t)|}$. For $t > \alpha$, $y'_\alpha(t) = 2(t - \alpha) =$

$2\sqrt{(t-\alpha)^2} = 2\sqrt{|y_\alpha(t)|}$. For $t = \alpha$,

$$\begin{aligned} \lim_{h\to 0^+} \frac{y_\alpha(\alpha+h) - y_\alpha(\alpha)}{h} &= \lim_{h\to 0^+} \frac{h^2}{h} \\ &= 0 \\ \lim_{h\to 0^-} \frac{y_\alpha(\alpha+h) - y_\alpha(\alpha)}{h} &= \lim_{h\to 0^-} \frac{0}{h} \\ &= 0 \end{aligned}$$

so $y'_\alpha(\alpha) = 0 = 2\sqrt{|y_\alpha(\alpha)|}$. Finally, $y_\alpha(0) = 0$, so y_α is a solution of the initial value problem (2), so we see the solution is decidedly not unique!

Remark 5 The initial value problem of Equation (1) has a solution defined on the interval

$$(\inf \{t : \forall s \in (t, t_0] \ (y(s), s) \in U\}, \sup \{t : \forall s \in [t_0, t) \ (y(s), s) \in U\})$$

and it is unique on that interval provided that F is locally Lipschitz on U, i.e. for every $(y,t) \in U$, there is an open set V with $(y,t) \in V \subseteq U$ such that F is Lipschitz on V.

2 Autonomous Differential Equations

In many situations of interest, the function F in the differential equation depends only on y and not on t.

Definition 6 An *autonomous differential equation* is a differential equation the form

$$y'(t) = F(y(t))$$

where $F : \mathbf{R}^n \to \mathbf{R}$ depends on t only through the value of $y(t)$. A *stationary point* of an autonomous differential equation is a point $y_s \in \mathbf{R}^n$ such that $F(y_s) = 0$

We can study the qualitative properties of solutions of autonomous differential equations by looking for stationary points. Note that the constant function

$$y(t) = y_s$$

is a solution of the initial value problem

$$y' = F(y), \; y(t_0) = y_s$$

If F satisfies a Lipschitz condition, the constant function is the unique solution of the initial value problem.

If F is C^2, then from Taylor's Theorem, we know near a stationary point y_s, we have

$$\begin{aligned} F(y_s + h) &= F(y_s) + DF(y_s)h + O\left(|h|^2\right) \\ &= DF(y_s)h + O\left(|h|^2\right) \end{aligned}$$

Thus, when we are sufficiently close to the stationary point, the solutions of the differential equation are closely approximated by the solutions of the *linear* differential equation

$$y' = (y - Y_s)' = DF(y_s)(y - y_s)$$

3 Complex Exponentials

The exponential function e^x (for $x \in \mathbf{R}$ or $x \in \mathbf{C}$) is given by the Taylor series

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

For $x, y \in \mathbf{C}$, we have

$$e^{x+y} = e^x e^y$$

If $x \in \mathbf{C}$, $x = a + ib$ for $a, b \in \mathbf{R}$, so

$$\begin{aligned} e^x &= e^{a+ib} \\ &= e^a e^{ib} \\ &= e^a \left(\sum_{k=0}^{\infty} \frac{(ib)^k}{k!} \right) \\ &= e^a \left(\sum_{k=0}^{\infty} \frac{(ib)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(ib)^{2k+1}}{(2k+1)!} \right) \\ &= e^a \left(\sum_{k=0}^{\infty} i^{2k} \frac{b^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} i^{2k} \frac{b^{2k+1}}{(2k+1)!} \right) \end{aligned}$$

$$
\begin{aligned}
&= e^a \left(\sum_{k=0}^{\infty} (-1)^k \frac{b^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{b^{2k+1}}{(2k+1)!} \right) \\
&= e^a (\cos b + i \sin b)
\end{aligned}
$$

Now suppose that $t \in \mathbf{R}$, so

$$e^{tx} = e^{ta+itb} = e^{ta}(\cos tb + i \sin tb)$$

- If $a < 0$, then $e^{tx} \to 0$ as $t \to \infty$
- If $a > 0$, then $|e^{tx}| \to \infty$ as $t \to \infty$
- If $a = 0$, then $|e^{tx}| = 1$ for all $t \in \mathbf{R}$

4 Homogenous Linear Differential Equations with Constant Coefficients

Let M be a constant $n \times n$ real matrix. Linear differential equations of the type

$$y' = (y - y_s)' = M(y - y_s) \tag{3}$$

are called *homogeneous linear differential equations with constant coefficients.*[1]

Differential Equation (3) has a complete solution in closed form. Note that the matrix representing $DF(y_s)$ need not be symmetric, so it need not be diagonalizable. However, if the $n \times n$ matrix $DF(y_s)$ *is* diagonalizable over $\mathbf{C}$, the complete solution takes the following particularly simple form:

Theorem 7 *Consider the linear differential equation*

$$y' = (y - y_s)' = M(y - y_s)$$

where M is a real $n \times n$ matrix. Suppose that M can be diagonalized over the complex field $\mathbf{C}$. *Let U be the standard basis of* $\mathbf{R}^n$ *and $V = \{v_1, \ldots, v_n\}$ be a*

[1]Later, we shall study equations of the type $y' = (y - y_s)' = M(y - y_s) + H(t)$, where $H : \mathbf{R} \to \mathbf{R}^n$; such equations are called *inhomogeneous*; homogenous equations are ones in which H is identically zero.

*basis of (complex) eigenvectors corresponding to the eigenvectors $\lambda_1, \ldots, \lambda_n \in$ **C**. Then the solution of the initial value problem is given by*

$$y(t) = y_s + (Mtx)_{U,V}(id) \begin{pmatrix} e^{\lambda_1(t-t_0)} & 0 & 0 & \cdots & 0 \\ 0 & e^{\lambda_2(t-t_0)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & e^{\lambda_n(t-t_0)} \end{pmatrix} (Mtx)_{V,U}(id)(y(t_0) - y_s) \tag{4}$$

and the general complex solution is obtained by allowing $y(t_0)$ to vary over $\mathbf{C}^n$; it has n complex degrees of freedom. The general real solution is obtained by allowing $y(t_0)$ to vary over $\mathbf{R}^n$; it has n real degrees of freedom. Every real solution is a linear combination of the real and imaginary parts of a complex solution. In particular,

1. *If the real part of each eigenvalue is less than zero, all solutions converge to y_s*

2. *If the real part of each eigenvalue is greater than zero, all solutions diverge from y_s and tend to infinity*

3. *If the real parts of some eigenvalues are less than zero and the real parts of other eigenvalues are greater than zero, solutions follow "hyperbolic" paths*

4. *If the real parts of all eigenvalues are zero, all solutions follow closed cycles around y_s*

Remark 8 If one or more of the eigenvalues are complex, each of the three matrices in Equation (4) will contain complex entries, but the *product* of the three matrices is *real*. Thus, if the initial condition y_0 is real, Equation (4) gives us a real solution; indeed, it gives us the unique solution of the initial value problem.

Remark 9 Given a fixed time t_0, the general real solution is obtained by varying the initial values of $y(t_0)$ over $\mathbf{R}^n$, which provides n real degrees of freedom. You might think that varying t_0 provides one additional degree of freedom, but it doesn't. Given any solution satisfying the initial condition $y(t_0) = y_0$, the solution is defined on some interval $(t_0 - \delta, t_0 + \delta)$; given $t_1 \in (t_0 - \delta, t_0 + \delta)$, let $y_1 = y(t_1)$; then the solution with initial condition $y(t_1) = y_1$ is the same as the solution with initial condition $y(t_0) = y_0$. The same holds true for the general complex solution.

Proof: We rewrite the differential equation in terms of a new variable

$$z = (Mtx)_{V,U}(id) \cdot y$$

which is the representation of the solution with respect to the basis V of eigenvectors. Let $z_s = (Mtx)_{V,U}(id) \cdot y_s$. Then we have

$$\begin{aligned} z - z_s &= (Mtx)_{V,U}(id) \cdot (y - y_s) \\ (z - z_s)' &= z' \\ &= (Mtx)_{V,U}(id) \cdot y' \\ &= (Mtx)_{V,U}(id) \cdot M \cdot (y - y_s) \\ &= (Mtx)_{V,U} \cdot M \cdot (Mtx)_{U,V} \cdot (z - z_s) \\ &= B(z - z_s) \end{aligned}$$

where

$$B = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Thus, the i^{th} component of $(z(t) - z_s)$ satisfies the differential equation

$$(z(t) - z_s)'_i = \lambda_i (z(t) - z_s)_i$$

so

$$(z(t) - z_s)_i = e^{\lambda_i (t - t_0)} (z(t_0) - z_s)_i$$

so

$z(t) - z_s$

$$= \begin{pmatrix} e^{\lambda_1(t-t_0)} & 0 & 0 & \cdots & 0 \\ 0 & e^{\lambda_2(t-t_0)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & e^{\lambda_n(t-t_0)} \end{pmatrix} \cdot (z(t_0) - z_s)$$

$$
\begin{aligned}
& y(t) - y_s \\
&= (Mtx)_{U,V}(id) \cdot (z(t) - z_s) \\
&= (Mtx)_{U,V}(id) \cdot \begin{pmatrix} e^{\lambda_1(t-t_0)} & 0 & 0 & \cdots & 0 \\ 0 & e^{\lambda_2(t-t_0)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & e^{\lambda_n(t-t_0)} \end{pmatrix} \cdot (z(t_0) - z_s) \\
&= (Mtx)_{U,V}(id) \cdot \begin{pmatrix} e^{\lambda_1(t-t_0)} & 0 & 0 & \cdots & 0 \\ 0 & e^{\lambda_2(t-t_0)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & e^{\lambda_n(t-t_0)} \end{pmatrix} \cdot (Mtx)_{V,U}(id) \cdot (y(t_0) - y_s)
\end{aligned}
$$

which shows that the solution to the initial value problem is as claimed; the general complex solution is found by varying $y(t_0)$ over $\mathbf{C}^n$. Given a differentiable function $y : [t_0, \infty) \to \mathbf{C}^n$, write $y(t) = w(t) + ix(t)$ as the sum of its real and imaginary parts. Then $y'(t) = w'(t) + ix'(t)$. Since the matrix M is real, y is a complex solution if and only if w and x are real solutions. A solution is real if and only if $y(t_0) \in \mathbf{R}^n$, so the general real solution has n real degrees of freedom; the solution of the initial value problem is determined from the n real initial conditions $y(t_0)$. Statements (1)-(4) about the asymptotic properties of the solutions follow from the properties of complex exponentials in Section 3.■

5 The Form of Real Solutions

When the eigenvalues of a real matrix M are complex, the corresponding eigenvectors must be complex also. Consequently, even for a real solution y of the undiagonalized equation, the solution z of the diagonalized equation, which is the representation of y in terms of the basis V of eigenvectors, will be complex. However, as noted in the previous section, the *product* of the three matrices in Equation (4) is real, and hence the equation gives us a real solution, indeed the unique solution of the initial value problem. The general solution can be obtained by varying the initial condition y_0.

We can determine the form of the real solutions once we know the eigenvalues, and in one important case, this provides us with an easier way to find the general solution of the differential equation, or the unique solution of the initial value problem.

Theorem 10 *Consider the differential equation*

$$y' = (y - y_s)' = M(y - y_s)$$

Suppose that the matrix M can be diagonalized over $\mathbf{C}$. *Let the eigenvalues of M be*

$$a_1 + ib_1, a_1 - ib_1, \ldots, a_m + ib_m, a_m - ib_m, a_{m+1}, \ldots, a_{n-m}$$

Then for each fixed $i = 1, \ldots, n$, every real solution is of the form

$$\begin{aligned} (y(t) - y_s)_i &= \sum_{j=1}^{m} e^{a_j(t-t_0)} \left(C_{ij} \cos b_j(t - t_0) + D_{ij} \sin b_j(t - t_0)\right) \\ &\quad + \sum_{j=m+1}^{n-m} C_{ij} e^{a_j(t-t_0)} \end{aligned}$$

The n^2 parameters

$$\{C_{ij} : i = 1, \ldots, n; j = 1, \ldots, n - m\} \cup \{D_{ij} : i = 1, \ldots, n; j = 1, \ldots, m\}$$

have n real degrees of freedom. The parameters are uniquely determined from the n real initial conditions of an Initial Value Problem.

Proof: Rewrite the expression for the solution y as

$$(y(t) - y_s)_i = \sum_{j=1}^{n} \gamma_{ij} e^{\lambda_j(t-t_0)}$$

Recall that the non-real eigenvalues occur in conjugate pairs, so suppose that

$$\lambda_j = a + ib, \ \lambda_k = a - ib$$

so the expression for $(y(t) - y_s)$ contains the pair of terms

$$\begin{aligned} &\gamma_{ij} e^{\lambda_j(t-t_0)} + \gamma_{ik} e^{\lambda_k(t-t_0)} \\ &= \gamma_{ij} e^{a(t-t_0)} \left(\cos b(t - t_0) + i \sin b(t - t_0)\right) \\ &\quad + \gamma_{ik} e^{a(t-t_0)} \left(\cos b(t - t_0) - i \sin b(t - t_0)\right) \\ &= e^{a(t-t_0)} \left((\gamma_{ij} + \gamma_{ik}) \cos b(t - t_0) + i\,(\gamma_{ij} - \gamma_{ik}) \sin b(t - t_0)\right) \\ &= e^{a(t-t_0)} \left(C_{ij} \cos b(t - t_0) + D_{ij} \sin b(t - t_0)\right) \end{aligned}$$

Since this must be real for all t, we must have

$$C_{ij} = \gamma_{ij} + \gamma_{ik} \in \mathbf{R} \text{ and } D_{ij} = i\left(\gamma_{ij} - \gamma_{ik}\right) \in \mathbf{R}$$

so γ_{ij} and γ_{ik} are complex conjugates; this can also be shown directly from the matrix formula for y in terms of z.

Thus, if the eigenvalues $\lambda_1, \ldots, \lambda_n$ are

$$a_1 + ib_1, a_1 - ib_1, a_2 + ib_2, a_2 - ib_2, \ldots,$$
$$a_m + ib_m, a_m - ib_m, a_{m+1}, \ldots, a_{n-m}$$

every real solution will be of the form

$$\begin{aligned}(y(t) - y_s)_i &= \sum_{j=1}^{m} e^{a_j(t-t_0)} \left(C_{ij} \cos b_j(t-t_0) + D_{ij} \sin b_j(t-t_0)\right) \\ &\quad + \sum_{j=m+1}^{n-m} C_{ij} e^{a_j(t-t_0)}\end{aligned}$$

Since the differential equation satisfies a Lipschitz condition, the Initial Value Problem has a unique solution determined by the n real initial conditions. Thus, the general solution has exactly n real degrees of freedom in the n^2 coefficients.■

Remark 11 The constraints among the coefficients C_{ij}, D_{ij} can be complicated. One cannot just solve for the coefficients of y_1 from the initial conditions, then derive the coefficients for $y_2, \ldots, y_n$. For example, consider the differential equation

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

The eigenvalues are 2 and 1. If we set

$$\begin{aligned} y_1(t) &= C_{11} e^{2(t-t_0)} + C_{12} e^{t-t_0} \\ y_2(t) &= C_{21} e^{2(t-t_0)} + C_{22} e^{t-t_0} \end{aligned}$$

we get

$$\begin{aligned} y_1(t_0) &= C_{11} + C_{12} \\ y_2(t_0) &= C_{21} + C_{22} \end{aligned}$$

which doesn't have a unique solution. However, from the original differential equation, we have

$$y_1(t) = y_1(t_0)e^{2(t-t_0)},\ y_2(t) = y_2(t_0)e^{t-t_0}$$

so

$$\begin{array}{cc} C_{11} = y_1(t_0) & C_{12} = 0 \\ C_{21} = 0 & C_{22} = y_2(t_0) \end{array}$$

One *can* find the solution to the Initial Value Problem by plugging the n real initial conditions into Equation (4), and the general solution by varying the initial conditions.

However, in the important special case

$$\bar{y} = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}$$

the coefficients

$$C_{11}, \ldots, C_{1(n-m)}, D_{11}, \ldots, D_{1m}$$

in the general solution are arbitrary real numbers; once they are set, the other coefficients are determined. Write

$$\begin{aligned} y(t) - y_s &= \sum_{j=1}^{m} e^{a_j(t-t_0)} \left(C_j \cos b_j(t-t_0) + D_j \sin b_j(t-t_0) \right) \\ &\quad + \sum_{j=m+1}^{n-m} C_j e^{a_j(t-t_0)} \end{aligned}$$

For the Initial Value Problem, compute the first $n-1$ derivatives of y at t_0 and set them equal to the initial conditions. This yield n linear equations in the n coefficients, which have a unique solution.

Note also that

$$\begin{aligned} C_j e^{a_j(t-t_0)} &= \left(C_j e^{-a_j t_0} \right) e^{a_j t} \\ \cos b_j(t-t_0) &= \cos\left(b_j t - b_j t_0 \right) \\ &= \cos b_j t \cos b_j t_0 + \sin b_j t \sin b_j t_0 \\ \sin b_j(t-t_0) &= \sin\left(b_j t - b_j t_0 \right) \\ &= -\cos b_j t \sin b_j t_0 + \sin b_j t \cos b_j t_0 \end{aligned}$$

so we can also write

$$\begin{aligned} y(t) - y_s &= \sum_{j=1}^{m} e^{a_j t} \left(C_j \cos b_j t + D_j \sin b_j t \right) \\ &\quad + \sum_{j=m+1}^{n-m} C_j e^{a_j t} \end{aligned}$$

6 Second Order Linear Differential Equations

Consider the second order differential equation $y'' = cy + by'$. Let

$$\bar{y}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

Then we have

$$\begin{aligned} \bar{y}'(t) &= \begin{pmatrix} y'(t) \\ y''(t) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ c & b \end{pmatrix} \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ c & b \end{pmatrix} \bar{y} \end{aligned}$$

The eigenvalues are $\frac{b \pm \sqrt{b^2+4c}}{2}$, the roots of the characteristic polynomial $\lambda^2 - b\lambda - c = 0$. We have the following cases:

1. $c = -\frac{b^2}{4}$: the eigenvalues are equal to $\frac{b}{2}$; the matrix may not be diagonalizable, so the solutions may have a more complicated form.

2. $c > -\frac{b^2}{4}$: $b^2 + 4c > 0$, so the eigenvalues are real and distinct, hence the matrix is diagonalizable.

   (a) $c > 0$: the eigenvalues are real and of opposite sign, so the solutions follow roughly hyperbolic trajectories. The asymptotes are the two eigenvectors; along one eigenvector, the solutions converge to zero, while along the other, they tend to infinity.

   (b) $c < 0$ the eigenvalues are real and of the same sign as b

i. $b > 0$: the solutions tend to infinity.

ii. $b < 0$: the solutions converge to zero.

iii. $b = 0$: impossible since $0 > c > -\frac{b^2}{4} > 0$.

(c) $c = 0$: one eigenvalue is 0, while the other equals b.

i. $b > 0$: the solutions tend to infinity along one eigenvector, and are constant along the other eigenvector.

ii. $b < 0$: the solutions tend to zero along one eigenvector, and are constant along the other eigenvector.

iii. $b = 0$: impossible since $0 = c > -\frac{b^2}{4} = 0$.

3. $c < -\frac{b^2}{4}$. In this case, the eigenvalues are complex conjugates whose real parts have the same sign as b.

(a) $b < 0$: the solutions spiral in toward 0

(b) $b > 0$: the solutions. spiral outward and tend to infinity.

(c) $b = 0$: the solutions are closed cycles around zero.

Example 12 Consider the second order linear differential equation

$$y'' = 2y + y'$$

We let

$$\bar{y} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

so the equation becomes

$$\bar{y}' = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \bar{y}$$

The eigenvalues are roots of the characteristic polynomial

$$\lambda^2 - \lambda - 2 = 0$$

so the eigenvalues and corresponding eigenvectors are given by

$$\begin{array}{ll} \lambda_1 = 2 & v_1 = (1, 2) \\ \lambda_2 = -1 & v_2 = (1, -1) \end{array}$$

From this information alone, we know the qualitative properties of the solutions are as given in the phase plane diagram:

- The solutions are roughly hyperbolic in shape with asymptotes along the eigenvectors. Along the eigenvector v_1, the solutions flow off to infinity; along the eigenvector v_2, the solutions flow to zero.
- The solutions flow in directions consistent with the flows along the asymptotes.
- On the y-axis, we have $y' = 0$, which means that everywhere on the y-axis (except at the stationary point 0), the solution must have a vertical tangent.
- On the y'-axis, we have $y = 0$, so we have

  $$y'' = 2y + y' = y'$$

  Thus, above the y-axis, $y'' = y' > 0$, so y' is increasing along the direction of the solution; below the y-axis, $y'' = y' < 0$, so y' is decreasing along the direction of the solution.
- Along the line $y' = -2y$, $y'' = 2y - 2y = 0$, so y' achieves a minimum or maximum where it crosses that line.

We seek to find the general solution. We have described two methods.

- This method does not depend on the special structure $\bar{y} = \begin{pmatrix} y \\ y' \end{pmatrix}$.

  ¿From Equation (4), the general solution is given by

$$\begin{aligned}
\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} &= Mtx_{U,V}(id) \begin{pmatrix} e^{2(t-t_0)} & 0 \\ 0 & e^{-(t-t_0)} \end{pmatrix} Mtx_{V,U}(id) \begin{pmatrix} y(t_0) \\ y'(t_0) \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} e^{2(t-t_0)} & 0 \\ 0 & e^{-(t-t_0)} \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{pmatrix} \begin{pmatrix} y(t_0) \\ y'(t_0) \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \frac{e^{2(t-t_0)}}{3} & \frac{e^{2(t-t_0)}}{3} \\ \frac{2e^{-(t-t_0)}}{3} & -\frac{e^{-(t-t_0)}}{3} \end{pmatrix} \begin{pmatrix} y(t_0) \\ y'(t_0) \end{pmatrix} \\
&= \begin{pmatrix} \frac{e^{2(t-t_0)}+2e^{-(t-t_0)}}{3} & \frac{e^{2(t-t_0)}-e^{-(t-t_0)}}{3} \\ \frac{2e^{2(t-t_0)}-2e^{-(t-t_0)}}{3} & \frac{2e^{2(t-t_0)}+e^{-(t-t_0)}}{3} \end{pmatrix} \begin{pmatrix} y(t_0) \\ y'(t_0) \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} \frac{y(t_0)+y'(t_0)}{3}e^{2(t-t_0)} + \frac{2y(t_0)-y'(t_0)}{3}e^{-(t-t_0)} \\ \\ \frac{2y(t_0)+2y'(t_0)}{3}e^{2(t-t_0)} + \frac{-2y(t_0)+y'(t_0)}{3}e^{-(t-t_0)} \end{pmatrix}$$

The general solution has two real degrees of freedom; a specific solution is determined by specifying the initial conditions $y(t_0)$ and $y'(t_0)$.

- The second method depends on the special structure

$$\bar{y} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

In this setting, it is easier to find the general solution by setting

$$y(t) = C_1 e^{2(t-t_0)} + C_2 e^{-(t-t_0)}$$

and solving for the coefficients C_1 and C_2:

$$\begin{aligned} y(t_0) &= C_1 + C_2 \\ y'(t_0) &= 2C_1 e^{2(t-t_0)} - C_2 e^{-(t-t_0)} \\ y'(t_0) &= 2C_1 - C_2 \\ C_1 &= \frac{y(t_0) + y'(t_0)}{3} \\ C_2 &= \frac{2y(t_0) - y'(t_0)}{3} \\ y(t) &= \frac{y(t_0) + y'(t_0)}{3} e^{2(t-t_0)} + \frac{2y(t_0) - y'(t_0)}{3} e^{-(t-t_0)} \end{aligned}$$

Typically, we are not interested in the solution for y'; however, we can always obtain it by differentiating the solution for y.

7 Inhomogeneous Linear Differential Equations, with Nonconstant Coefficients

Consider the inhomogeneous linear differential equation

$$y' = M(t)y + H(t) \tag{5}$$

Here, M is a continuous function from t to the set of $n \times n$ matrices and H is a continuous function from t to $\mathbf{R}^n$. We first show that there is a close relationship between solutions of the inhomogeneous linear differential equation (5) and the associated homogeneous linear differential equation

$$y' = M(t)y \tag{6}$$

Theorem 13 *The general solution of the inhomogeneous linear differential equation (5) is*

$$y_h + y_p$$

where y_h is the general solution of the homogeneous linear differential equation (6) and y_p is any particular solution of the inhomogeneous linear differential equation (5).

Proof: Fix any particular solution y_p of the inhomogeneous differential equation (5). Suppose that y_h is any solution of the corresponding homogeneous differential equation (6). Let $y_i(t) = y_h(t) + y_p(t)$. Then

$$\begin{aligned} y_i'(t) &= y_h'(t) + y_p'(t) \\ &= M(t)y_h(t) + M(t)y_p(t) + H(t) \\ &= M(t)(y_h(t) + y_p(t)) + H(t) \\ &= M(t)y_i(t) + H(t) \end{aligned}$$

so y_i is a solution of the inhomogeneous differential equation (5).

Conversely, suppose that y_i is any solutions of the inhomogenous differential equation (5). Let $y_h(t) = y_i(t) - y_p(t)$. Then

$$\begin{aligned} y_h'(t) &= y_p'(t) - y_i'(t) \\ &= M(t)y_i(t) + H(t) - M(t)y_p(t) - H(t) \\ &= M(t)(y_i(t) - y_p(t)) \\ &= M(t)y_h(t) \end{aligned}$$

so y_h is a solution of the homogeneous differential equation (6) and $y_i = y + h + y_p$.■

Thus, in order to find the general solution of the inhomogeneous equation, one needs to find the general solution of the homogeneous equation and a particular solution of the inhomogeneous equation.

Here for an $n \times n$ matrix M, we define

$$e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!} = I + M + \frac{M^2}{2} + \cdots$$

and

$$e^{tM} = \sum_{k=0}^{\infty} \frac{t^k M^k}{k!}$$

A particular solution of the inhomogeneous equation in the constant coefficient case has the following simple form:

Theorem 14 *Consider the inhomogeneous linear differential equation (5), and suppose that $M(t)$ is a constant matrix M, independent of t. A particular solution of the inhomogeneous linear differential equation (5), satisfying the initial condition $y_p(t_0) = y_0$, is given by*

$$y_p(t) = e^{(t-t_0)M} y_0 + \int_{t_0}^{t} e^{(t-s)M} H(s)\, ds \qquad (7)$$

Proof: We verify that y_p solves (7):

$$\begin{aligned}
y_p(t) &= e^{(t-t_0)M} y_0 + \int_{t_0}^{t} e^{(t-s)M} H(s)\, ds \\
&= e^{(t-t_0)M} y_0 + \int_{t_0}^{t} e^{(t-t_0)M} e^{-(s-t_0)M} H(s)\, ds \\
&= e^{(t-t_0)M} \left(y_0 + \int_{t_0}^{t} e^{-(s-t_0)M} H(s)\, ds \right) \\
y_p'(t) &= M e^{(t-t_0)M} \left(y_0 + \int_{t_0}^{t} e^{-(s-t_0)M} H(s)\, ds \right) \\
&\quad + e^{(t-t_0)M} \left(e^{-(t_0-t)M} H(t) \right) \\
&= M y_p(t) + H(t) \\
y_p(t_0) &= e^{(t_0-t_0)M} y_0 + \int_{t_0}^{t_0} e^{(s-t_0)M} H(s)\, ds \\
&= y_0
\end{aligned}$$

■

Example 15

Consider the inhomogeneous linear differential equation

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

By Theorem 14, a particular solution is given by

$$\begin{aligned}
y_p(t) &= e^{(t-t_0)M} y_0 + \int_{t_0}^{t} e^{(t-s)M} H(s)\, ds \\
&= \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} e^{(t-s)} & 0 \\ 0 & e^{-(t-s)} \end{pmatrix} \begin{pmatrix} \sin s \\ \cos s \end{pmatrix} ds \\
&= \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix} + \int_0^t \begin{pmatrix} e^{t-s} \sin s \\ e^{s-t} \cos s \end{pmatrix} ds \\
&= \begin{pmatrix} e^t \left(1 + \int_0^t e^{-s} \sin s\, ds\right) \\ e^{-t} \left(1 + \int_0^t e^{s} \cos s\, ds\right) \end{pmatrix} \\
\int_0^t e^{-s} \sin s\, ds &= -e^{-s} \sin s \Big|_0^t - \int_0^t -e^{-s} \cos s\, ds \\
&= -e^{-t} \sin t + e^0 \sin 0 + \int_0^t e^{-s} \cos s\, ds \\
&= -e^{-t} \sin t + -e^{-s} \cos s \Big|_0^t - \int_0^t -e^{-s}(-\sin s)\, ds \\
&= -e^{-t} \sin t + -e^{-t} \cos t + e^0 \cos 0 - \int_0^t e^{-s} \sin s\, ds \\
&= -e^{-t}(\sin t + \cos t) + 1 - \int_0^t e^{-s} \sin s\, ds \\
2\int_0^t e^{-s} \sin s\, ds &= -e^{-t}(\sin t + \cos t) + 1 \\
\int_0^t e^{-s} \sin s\, ds &= \frac{-e^{-t}(\sin t + \cos t) + 1}{2} \\
\int_0^t e^{s} \cos s\, ds &= e^s \cos s|_0^t - \int_0^t e^s(-\sin s)\, ds \\
&= e^t \cos t - e^0 \cos 0 + \int_0^t e^s \sin s\, ds \\
&= e^t \cos t - 1 + e^s \sin s|_0^t - \int_0^t e^s \cos s\, ds
\end{aligned}$$

$$
\begin{aligned}
&= e^t\cos t - 1 + e^t\sin t + e^0\sin 0 - \int_0^t e^s\cos s\,ds \\
&= e^t(\sin t + \cos t) - 1 - \int_0^t e^s\cos s\,ds
\end{aligned}
$$

$$
\begin{aligned}
2\int_0^t e^s\cos s\,ds &= e^t(\sin t + \cos t) - 1 \\
\int_0^t e^s\cos s\,ds &= \frac{e^t(\sin t + \cos t) - 1}{2}
\end{aligned}
$$

$$
\begin{aligned}
y_p(t) &= \begin{pmatrix} e^t\left(1 + \int_0^t e^{-s}\sin s\,ds\right) \\ e^{-t}\left(1 + \int_0^t e^{s}\cos s\,ds\right) \end{pmatrix} \\
&= \begin{pmatrix} e^t\left(1 + \frac{-e^{-t}(\sin t+\cos t)+1}{2}\right) \\ e^{-t}\left(1 + \frac{e^{t}(\sin t+\cos t)-1}{2}\right) \end{pmatrix} \\
&= \begin{pmatrix} e^t\left(\frac{3-e^{-t}(\sin t+\cos t)}{2}\right) \\ e^{-t}\left(\frac{1+e^{t}(\sin t+\cos t)}{2}\right) \end{pmatrix} \\
&= \begin{pmatrix} \frac{3e^t-\sin t-\cos t)}{2} \\ \frac{e^{-t}+\sin t+\cos t}{2} \end{pmatrix}
\end{aligned}
$$

Thus, the general solution of the original inhomogeneous equation is given by

$$
\begin{aligned}
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} C_1e^t \\ C_2e^{-t} \end{pmatrix} + \begin{pmatrix} \frac{3e^t-\sin t-\cos t}{2} \\ \frac{e^{-t}+\sin t+\cos t}{2} \end{pmatrix} \\
&= \begin{pmatrix} D_1e^t - \frac{\sin t+\cos t}{2} \\ D_2e^{-t} + \frac{\sin t+\cos t}{2} \end{pmatrix}
\end{aligned}
$$

where D_1 and D_2 are arbitrary real constants.

8 Nonlinear Differential Equations: Linearization

Nonlinear differential equations may be very difficult to solve in closed form. There are a number of specific techniques that solve special classes of equations, and there are numerical methods to compute numerical solutions of any ordinary differential equation. Unfortunately, time does not permit us to discuss either these special classes of equations, or numerical methods.

Linearization allows us to obtain qualitative information about the solutions of nonlinear autonomous equations. As we noted in Section 2, the idea is to find stationary points of the equation, then study the solutions of the linearized equation near the stationary points. This gives us a reasonably, but not completely, reliable to the behavior of the solutions of the original equation.

Example 16 Consider the second-order nonlinear autonomous differential equation

$$y'' = -\alpha^2 \sin y, \ \alpha > 0$$

This is the equation of motion of a pendulum, but it has much in common with all cyclical processes, including processes such as business cycles. The equation is very difficult to solve because of the nonlinearity of the function $\sin y$. We define

$$\bar{y}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

so that the differential equation becomes

$$\bar{y}'(t) = \begin{pmatrix} y_2(t) \\ -\alpha^2 \sin y_1(t) \end{pmatrix}$$

Let

$$F(\bar{y}) = \begin{pmatrix} y_2(t) \\ -\alpha^2 \sin y_1(t) \end{pmatrix}$$

We solve for the points $\bar{y}$ such that $F(\bar{y}) = 0$:

$$\begin{aligned} F(\bar{y}) = 0 \quad &\Rightarrow \quad \begin{pmatrix} y_2(t) \\ -\alpha^2 \sin y_1(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\Rightarrow \quad \sin y_1 = 0 \text{ and } y_2 = 0 \\ &\Rightarrow \quad y_1 = n\pi \text{ and } y_2 = 0 \end{aligned}$$

so the set of stationary points is

$$\{(n\pi, 0) : n \in \mathbf{Z}\}$$

We linearize the equation around each of the stationary points. In other words, we take the first order Taylor polynomial for F:

$$\begin{aligned}
& F(n\pi + h, 0 + k) + o(|h| + |k|) \\
= \quad & F(n\pi, 0) + \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{pmatrix} \begin{pmatrix} h \\ \\ k \end{pmatrix} \\
= \quad & \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -\alpha^2 \cos n\pi & 0 \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} \\
= \quad & \begin{pmatrix} 0 & 1 \\ (-1)^{n+1}\alpha^2 & 0 \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix}
\end{aligned}$$

For n even, the eigenvalues are

$$\begin{aligned}
\lambda^2 + \alpha^2 &= 0 \\
\lambda_1 &= i\alpha, \ \lambda_2 = -i\alpha
\end{aligned}$$

Close to $(n\pi, 0)$ for n even, the solutions spiral around the stationary point. For $y_2 = y_1' > 0$, y_1 is increasing, so the solutions move in a clockwise direction.

For n odd, the eigenvalues and eigenvectors are

$$\begin{aligned}
\lambda^2 - \alpha^2 &= 0 \\
\lambda_1 &= \alpha, \ \lambda_2 = -\alpha \\
v_1 &= (1, \alpha), \ v_2 = (1, -\alpha)
\end{aligned}$$

Close to $(n\pi, 0)$ for n odd, the solutions are roughly hyperbolic; along v_2, they converge to the stationary point, while along v_1, they diverge from the stationary point.

From this information alone, we know the qualitative properties of the solutions are as given in the phase plane diagram on the next page:

- The solutions flow in directions consistent with the flows along the asymptotes.

- On the y-axis, we have $y' = 0$, which means that everywhere on the y-axis (except at the stationary points), the solution must have a vertical tangent.

- For $y = n\pi$, we have
$$y'' = -\alpha^2 \sin y = 0$$
so the derivative of y' is zero, so the tangent to the curve is horizontal.

If the initial value of $|y_2|$ is sufficiently large, the solutions no follow longer closed curves; this corresponds to the pendulum going "over the top" rather than oscillating back and forth.

9 Nonlinear Differential Equations–Stability

As we saw in the last section, linearization provides information about the qualitative properties of solutions of nonlinear differential equations near the stationary points. Suppose that y_s is a stationary point, and that the eigenvalues of the linearized equation at y_s all have strictly negative real parts. Then there exists $\varepsilon > 0$ such that, if $|y(0) - y_s| < \varepsilon$, then $\lim_{t\to\infty} y(t) = y_s$; all solutions of the original nonlinear equation which start sufficiently close to the stationary point y_s converge to y_s. On the other hand, if the egivenvalues of the linearized equation all have strictly positive real parts, then no solutions of the original nonlinear differential equation converge to y_s.

If the eigenvalues of the linearized equation at the stationary point y_s all have real part zero, then the solutions of the linearized equation are closed curves around y_s. In this case, the linearized equation tells us little about the solutions of the nonlinear equation. They may follow closed curves around y_s, converge to y_s, converge to a limit closed curve around y_s, diverge from y_s, or converge to y_s along certain directions and diverge from y_s along other directions.

In this section, we illustrate techniques to analyze the asymptotic behavior of solutions of the linearized equation when the eigenvalues all have real part zero.

Example 17 Consider the initial value problem

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} -9y_2(t) + 4y_1^3(t) + 4y_1(t)y_2^2(t) \\ 4y_1(t) + 9y_1^2(t)y_2(t) + 9y_2^3(t) \end{pmatrix}, \ y_1(0) = 3, \ y_2(0) = 0 \quad (8)$$

$y_s = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a stationary point. The linearization around this stationary point is

$$y'(t) = \begin{pmatrix} 0 & -9 \\ 4 & 0 \end{pmatrix} y$$

The characteristic equation is $\lambda^2 + 36 = 0$, so the matrix has distinct eigenvalues $\lambda_1 = 6i$ and $\lambda_2 = -6i$; since both of these have real part zero, we know the solutions of the linearized differential equation follows closed curves around zero. The associated eigenvectors are $v_1 = \begin{pmatrix} 3i/2 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -3i/2 \\ 1 \end{pmatrix}$, so the change of basis matrices are

$$Mtx_{U,V}(id) = \begin{pmatrix} 3i/2 & -3i/2 \\ 1 & 1 \end{pmatrix} \text{ and } Mtx_{V,U}(id) = \begin{pmatrix} -i/3 & 1/2 \\ i/3 & 1/2 \end{pmatrix}$$

Then the solution of the linearized initial value problem is

$$\begin{aligned}
y &= \begin{pmatrix} 3i/2 & -3i/2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{6ti} & 0 \\ 0 & e^{-6ti} \end{pmatrix} \begin{pmatrix} -i/3 & 1/2 \\ i/3 & 1/2 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 3i/2 & -3i/2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -ie^{6ti}/3 & e^{6ti}/2 \\ ie^{-6ti}/3 & e^{-6ti}/2 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} (e^{6ti} + e^{-6ti})/2 & (e^{6ti} - e^{-6ti})3i/4 \\ (e^{-6ti} - e^{6ti})i/3 & (e^{6ti} + e^{-6ti})/2 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \cos 6t & -3(\sin 6t)/2 \\ 2(\sin 6t)/3 & \cos 6t \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 3\cos 6t \\ 2\sin 6t \end{pmatrix}
\end{aligned}$$

since

$$\begin{aligned}
e^{6ti} + e^{-6ti} &= \cos 6t + i\sin 6t + \cos(-6t) + i\sin(-6t) \\
&= \cos 6t + i\sin 6t + \cos 6t - i\sin 6t \\
&= 2\cos 6t \\
e^{6ti} - e^{-6ti} &= \cos 6t + i\sin 6t - \cos(-6t) - i\sin(-6t) \\
&= \cos 6t + i\sin 6t - \cos 6t + i\sin 6t \\
&= 2i\sin 6t
\end{aligned}$$

Notice that

$$\begin{aligned}
\frac{y_1^2(t)}{9} + \frac{y_2^2(t)}{4} &= \frac{9\cos^2 t}{9} + \frac{4\sin^2(t)}{4} \\
&= \cos^2 t + \sin^2 t \\
&= 1
\end{aligned}$$

so the solution of the linearized initial value problem is a closed curve running counterclockwise around the ellipse with principal axes along the y_1 and y_2 axes, of length 3 and 2 respectively.

Let

$$G(y) = \frac{y_1^2}{9} + \frac{y_2^2}{4}$$

so that the solution runs around a level set of G. We can get more information about the asymptotic behavior of the solution of the orginal, unlinearized, initial value problem (8) by computing $\frac{dG(y(t))}{dt}$:

$$\begin{aligned}
\frac{dG(y(t))}{dt} &= \begin{pmatrix} \frac{\partial G}{\partial y_1} & \frac{\partial G}{\partial y_2} \end{pmatrix} \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} \\
&= \begin{pmatrix} \frac{2y_1(t)}{9} & \frac{y_2(t)}{2} \end{pmatrix} \begin{pmatrix} -9y_2(t) + 4y_1^3(t) + 4y_1(t)y_2^2(t) \\ 4y_1(t) + 9y_1^2(t)y_2(t) + 9y_2^3(t) \end{pmatrix} \\
&= -2y_1(t)y_2(t) + 8y_1^4(t)/9 + 8y_1^2(t)y_2^2(t)/9 \\
&\quad +2y_1(t)y_2(t) + 9y_1^2(t)y_2^2(t)/2 + 9y_2^4(t)/2 \\
&= 8y_1^4(t)/9 + 97y_1^2(t)y_2^2(t)/18 + 9y_2^4(t)/2 \\
&> 0 \text{ unless } y(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}$$

$y'(t)$ is tangent to the solution at every t, and $y'(t)$ always points outside the level curve of G through $y(t)$, as in the green arrows in the diagram. Thus, the solution of the initial value problem (8) spirals outward, always moving to higher level curves of G. Indeed, for $G(y) \geq 1$ (i.e., outside the ellipse which the solution of the linearized initial value problem falls), it is easy to see that

$$8y_1^4(t)/9 + 97y_1^2(t)y_2^2(t)/18 + 9y_2^4(t)/2 > \frac{8}{9}\left(y_1^2(t) + y_2^2(t)\right)^2$$

so $\frac{dG(y(t))}{dt}$ is uniformly bounded away from zero, so $G(y(t)) = \int_0^t \frac{dG(y(s))}{ds}\, ds \to \infty$ as $t \to \infty$. The linear terms become dwarfed by the higher order terms, which will determine whether the solution continues to spiral as it heads off into the distance.

Suppose now we consider instead the initial value problem

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} -9y_2(t) - 4y_1^3(t) - 4y_1(t)y_2^2(t) \\ 4y_1(t) - 9y_1^2(t)y_2(t) - 9y_2^3(t) \end{pmatrix}, \; y_1(0) = 3, \; y_2(0) = 0 \quad (9)$$

The linearized initial value problem has not changed. As before, we compute

$$\begin{aligned} \frac{dG(y(t))}{dt} &= \begin{pmatrix} \frac{\partial G}{\partial y_1} & \frac{\partial G}{\partial y_2} \end{pmatrix} \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} \\ &= \begin{pmatrix} \frac{2y_1(t)}{9} & \frac{y_2(t)}{2} \end{pmatrix} \begin{pmatrix} -9y_2(t) - 4y_1^3(t) - 4y_1(t)y_2^2(t) \\ 4y_1(t) - 9y_1^2(t)y_2(t) - 9y_2^3(t) \end{pmatrix} \\ &= -2y_1(t)y_2(t) - 8y_1^4(t)/9 - 8y_1^2(t)y_2^2(t)/9 \\ &\quad +2y_1(t)y_2(t) - 9y_1^2(t)y_2^2(t)/2 - 9y_2^4(t)/2 \\ &= -8y_1^4(t)/9 - 97y_1^2(t)y_2^2(t)/18 - 9y_2^4(t)/2 \\ &< 0 \text{ unless } y(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

As before, $y'(t)$ is tangent to the solution at every t, and $y'(t)$ always points inside the level curve of G through $y(t)$, as in the blue arrows in the diagram. Since $\frac{dG(y(t))}{dt} < 0$ except at the origin, then for all $C > 0$,

$$\alpha = \inf \left\{ \frac{dG(y(t))}{dt} : C \le G(y(t)) \le G(y(0)) \right\} < 0$$

since the region $\{y : C \le G(y) \le G(y(0))\}$ is compact. If $G(y(t)) \ge C$ for all t,

$$\begin{aligned} G(y(t)) &= G(y(0)) + \int_0^t \frac{dG(y(s))}{ds}\, ds \\ &\le G(y(0)) + \alpha t \\ &\to -\infty \text{ as } t \to \infty \end{aligned}$$

which is a contradiction. Thus, $G(y(t)) \to 0$ and the solution of the initial value problem (9) converges to the stationary point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to \infty$.

Example 18 This example was assigned as a problem in 2007, much to the regret of the students and the professor; the latter did penance by writing out the solution. The principal axes of elliptical solutions of the linearized equation align with the eigenvectors. In Example 17, the eigenvectors aligned with the standard basis. When the eigenvectors do not align with the standard basis, the algebraic computations become quite challenging and tedious to do by hand. This example shows that they can in fact be done by hand; with an appropriate symbolic manipulation computer program, they would be straightforward.

Consider the initial value problem

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} 5y_1(t) - 13y_2(t) - y_1^3(t) - y_1(t)y_2^2(t) \\ 13y_1(t) - 5y_2(t) - y_1^2(t)y_2(t) - y_2^3(t) \end{pmatrix}, \ y_1(0) = 3, \ y_2(0) = 3 \tag{10}$$

The linearization around the stationary point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is

$$y'(t) = \begin{pmatrix} 5 & -13 \\ 13 & -5 \end{pmatrix} y$$

The characteristic equation is $(5-\lambda)(-5-\lambda)+169 = 0$, $\lambda^2+144 = 0$, so there are distinct eigenvalues $\lambda_1 = 12i$ and $\lambda_2 = -12i$; since both of these have real part zero, we know the solution of the linearized initial value problem follows a closed curve around zero. The associated eigenvectors are $v_1 = \begin{pmatrix} 1 \\ \frac{5-12i}{13} \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ \frac{5+12i}{13} \end{pmatrix}$, so the change of basis matrices are

$$Mtx_{U,V}(id) = \begin{pmatrix} 1 & 1 \\ \frac{5-12i}{13} & \frac{5+12i}{13} \end{pmatrix} \text{ and } Mtx_{V,U}(id) = \begin{pmatrix} \frac{12-5i}{24} & \frac{13i}{24} \\ \frac{12+5i}{24} & -\frac{13i}{24} \end{pmatrix}$$

Then the solution of the linearized initial value problem is

$$\begin{aligned} y &= \begin{pmatrix} 1 & 1 \\ \frac{5-12i}{13} & \frac{5+12i}{13} \end{pmatrix} \begin{pmatrix} e^{12ti} & 0 \\ 0 & e^{-12ti} \end{pmatrix} \begin{pmatrix} \frac{12-5i}{24} & \frac{13i}{24} \\ \frac{12+5i}{24} & -\frac{13i}{24} \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ \frac{5-12i}{13} & \frac{5+12i}{13} \end{pmatrix} \begin{pmatrix} (12-5i)e^{12ti}/24 & 13ie^{12ti}/24 \\ (12+5i)e^{-12ti}/24 & -13ie^{-12ti}/24 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} (e^{12ti}+e^{-12ti})/2-5i(e^{12ti}-e^{-12ti})/24 & 13i(e^{12ti}-e^{-12ti})/24 \\ -13i((e^{12ti}-e^{-12ti})/24 & (e^{12ti}+e^{-12ti})/2+5i(e^{12ti}-e^{-12ti})/24 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} \cos 12t + 5(\sin 12t)/12 & -13(\sin 12t)/12 \\ 13(\sin 12t)/12 & \cos 12t - 5(\sin 12t)/12 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} 3\cos 12t - 2\sin 12t \\ 3\cos 12t + 2\sin 12t \end{pmatrix}
\end{aligned}
$$

since

$$
\begin{aligned}
e^{12ti}+e^{-12ti} &= \cos 12t + i\sin 12t + \cos(-12t) + i\sin(-12t) \\
&= \cos 12t + i\sin 12t + \cos 12t - i\sin 12t \\
&= 2\cos 12t \\
e^{12ti}-e^{-12ti} &= \cos 12t + i\sin 12t - \cos(-12t) - i\sin(-12t) \\
&= \cos 12t + i\sin 12t - \cos 12t + i\sin 12t \\
&= 2i\sin 12t
\end{aligned}
$$

Let

$$
\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} y_1(t)+y_2(t) \\ y_2(t)-y_1(t) \end{pmatrix} = \begin{pmatrix} 6\cos 12t \\ 4\sin 12t \end{pmatrix}
$$

so

$$
\frac{z_1^2(t)}{36} + \frac{z_2^2(t)}{16} = \cos^2 12t + \sin^2 12t = 1
$$

This shows that the solution of the linearized initial value problem follows an ellipse with principal axes of length $3\sqrt{2}$ and $2\sqrt{2}$ along the vectors $(1,1)$ and $(-1,1)$ counterclockwise.

In order to determine the asymptotic behavior of solutions of the nonlinear equation, we need to find a function G whose level sets are the solutions of the linearized equation. Let

$$
\begin{aligned}
G(y) &= 144\left(\frac{z_1^2}{36}+\frac{z_2^2}{16}\right) \\
&= 144\left(\frac{y_1^2+2y_1y_2+y_2^2}{36}+\frac{y_1^2-2y_1y_2+y_2^2}{16}\right) \\
&= 13y_1^2 - 10y_1y_2 + 13y_2^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{dG(y(t))}{dt} &= \begin{pmatrix} 26y_1(t) - 10y_2(t) & 26y_2(t) - 10y_1(t) \end{pmatrix} \begin{pmatrix} 5y_1(t) - 13y_2(t) - y_1^3(t) - y_1(t)y_2^2(t) \\ 13y_1(t) - 5y_2(t) - y_1^2(t)y_2(t) - y_2^3(t) \end{pmatrix} \\
&= 130y_1^2(t) - 338y_1(t)y_2(t) - 26y_1^4(t) - 26y_1^2(t)y_2^2(t) \\
&\quad -50y_1(t)y_2(t) + 130y_2^2(t) + 10y_2(t)y_1^3(t) + 10y_1(t)y_2^3(t) \\
&\quad +338y_1(t)y_2(t) - 130y_2^2(t) - 26y_1^2(t)y_2^2(t) - 26y_2^4(t) \\
&\quad -130y_1^2(t) + 50y_1(t)y_2(t) + 10y_1^3(t)y_2(t) + 10y_1(t)y_2^3(t) \\
&= -26(y_1^4(t) + 2y_1^2(t)y_2^2(t) + y_2^4(t)) \\
&\quad +20y_1(t)y_2(t)(y_1^2(t) + y_2^2(t)) \\
&= -26(y_1^2(t) + y_2^2(t))^2 + 20y_1(t)y_2(t)(y_1^2(t) + y_2^2(t)) \\
&= -16(y_1^2(t) + y_2^2(t))^2 - 10(y_1^2(t) + y_2^2(t))(y_1^2(t) - 2y_1(t)y_2(t) + y_2^2(t)) \\
&= -16(y_1^2(t) + y_2^2(t))^2 - 10(y_1^2(t) + y_2^2(t))(y_1(t) - y_2(t))^2 \\
&< 0 \text{ unless } y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

Therefore, the tangent to the solution of the nonlinear equation always points inward to the elliptical level sets of the linearized equation and, as in Example 17, the solution of the initial value problem (10) converges to the stationary point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to \infty$.

In the initial value problems (8), (9) and (10), we were lucky to some extent. We took G to be the function whose level sets are the solutions of the linearized differential equation, and found that the tangent to the solution always pointed outside the level curve in (8) and always pointed inside the level curve in (9). It is not hard to construct examples in which the tangent points outward at some points and inward at others, so the value $G(y(t))$ is not monotonic. One may be able to show by calculation that $G(y(t)) \to \infty$, so the solution disappears off into the distance, or that $G(y(t)) \to 0$, so the solution converges to the stationary point. An alternative method is to choose a *different* function G, whose level sets are not the solutions of the linearized differential equation, but for which one can prove that $\frac{dG(y(t))}{dt}$ is always positive or always negative; this is called Liapunov's Second Method (see Ritger and Rose [2]).

References

[1] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[2] Paul D. Ritger and Nicholas J. Rose, *Differential Equations with Applications*, McGraw-Hill, New York, 1968.