Econ 204 2016

Lecture 5

Outline

1. Properties of Real Functions (Sect. 2.6, cont.)
2. Monotonic Functions
3. Cauchy Sequences and Complete Metric Spaces
4. Contraction Mappings
5. Contraction Mapping Theorem

Announcements:
- PSI 1 due now
- Suggested solutions posted
- Due today

- PS 2 posted
- Due Tues 8/2
Examples in which a function $f$ is 1-1, onto, and continuous, but $g = f^{-1}$ is not continuous:

1. $f: [0,1] \cup (2,3] \to [0,2]$ given by
   
   $f(x) = \begin{cases} 
   x & \text{if } 0 \leq x \leq 1 \\
   x - 1 & \text{if } 2 < x \leq 3
   \end{cases}$

   Then
   
   $g = f^{-1}(y) = \begin{cases} 
   y & \text{if } 0 \leq y \leq 1 \\
   y + 1 & \text{if } 1 < y \leq 2
   \end{cases}$

2. $S' = \{ x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1 \}$

   $f: [0,2\pi) \to S'$ given by
   
   $f(\Theta) = (\cos \Theta, \sin \Theta)$
Properties of Real Functions

Here we first study properties of functions from \( \mathbb{R} \) to \( \mathbb{R} \), making use of the additional structure we have in \( \mathbb{R} \) as opposed to general metric spaces.

Let \( f : X \to \mathbb{R} \) where \( X \subseteq \mathbb{R} \). We say \( f \) is **bounded above** if

\[
f(X) = \{ r \in \mathbb{R} : f(x) = r \text{ for some } x \in X \}
\]

is bounded above. Similarly, we say \( f \) is **bounded below** if \( f(X) \) is bounded below. Finally, \( f \) is **bounded** if \( f \) is both bounded above and bounded below, that is, if \( f(X) \) is a bounded set.
Extreme Value Theorem

**Theorem 1** (Thm. 6.23, Extreme Value Theorem). Let $a, b \in \mathbb{R}$ with $a \leq b$ and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then $f$ assumes its minimum and maximum on $[a, b]$. That is, if

$$M = \sup_{t \in [a, b]} f(t), \quad m = \inf_{t \in [a, b]} f(t)$$

then $\exists t_M, t_m \in [a, b]$ such that $f(t_M) = M$ and $f(t_m) = m$.

*Proof.* Let

$$M = \sup \{ f(t) : t \in [a, b] \}$$

If $M$ is finite, then for each $n$, we may choose $t_n \in [a, b]$ such that $M \geq f(t_n) \geq M - \frac{1}{n}$ (if we couldn’t make such a choice, then $M - \frac{1}{n}$ would be an upper bound and $M$ would not be the
supremum). If $M$ is infinite, choose $t_n$ such that $f(t_n) \geq n$. By the Bolzano-Weierstrass Theorem, \{t_n\} contains a convergent subsequence \{t_{n_k}\}, with

$$\lim_{k \to \infty} t_{n_k} = t_0 \in [a, b]$$

Since $f$ is continuous,

$$f(t_0) = \lim_{t \to t_0} f(t) = \lim_{k \to \infty} f(t_{n_k}) = M$$

so $M$ is finite and

$$f(t_0) = M = \sup\{f(t) : t \in [a, b]\}$$

so $f$ attains its maximum and is bounded above.

The argument for the minimum is similar. \qed
Intermediate Value Theorem Redux

Theorem 2 (Thm. 6.24, Intermediate Value Theorem). Suppose \( f : [a, b] \to \mathbb{R} \) is continuous, and \( f(a) < d < f(b) \). Then there exists \( c \in (a, b) \) such that \( f(c) = d \).

Proof. Let

\[
B = \{ t \in [a, b] : f(t) < d \}
\]

\( a \in B \), so \( B \neq \emptyset \). By the Supremum Property, \( \sup B \) exists and is real so let \( c = \sup B \). Since \( a \in B \), \( c \geq a \). \( B \subseteq [a, b] \), so \( c \leq b \). Therefore, \( c \in [a, b] \). We claim that \( f(c) = d \).

Let

\[
t_n = \min \left\{ c + \frac{1}{n}, b \right\} \geq c
\]
Either \( t_n > c \), in which case \( t_n \notin B \), or \( t_n = c \), in which case \( t_n = b \) so \( f(t_n) > d \), so again \( t_n \notin B \); in either case, \( f(t_n) \geq d \).

Since \( f \) is continuous at \( c \), \( f(c) = \lim_{n \to \infty} f(t_n) \geq d \) (Theorem 3.5 in de la Fuente).

Since \( c = \sup B \), we may find \( s_n \in B \) such that
\[
    c \geq s_n \geq c - \frac{1}{n}
\]
\( \forall n > 0 \)

Since \( s_n \in B \), \( f(s_n) < d \). Since \( f \) is continuous at \( c \), \( f(c) = \lim_{n \to \infty} f(s_n) \leq d \) (Theorem 3.5 in de la Fuente).

Since \( d \leq f(c) \leq d \), \( f(c) = d \). Since \( f(a) < d \) and \( f(b) > d \), \( a \neq c \neq b \), so \( c \in (a, b) \).

\( \square \)
Monotonic Functions

**Definition 1.** A function $f : \mathbb{R} \to \mathbb{R}$ is monotonically increasing if

$$y > x \Rightarrow f(y) \geq f(x)$$

Monotonic functions are very well-behaved...
Monotonic Functions

**Theorem 3** (Thm. 6.27). Let $a, b \in \mathbb{R}$ with $a < b$, and let $f : (a, b) \to \mathbb{R}$ be monotonically increasing. Then the one-sided limits

- **right-hand limit**
  \[ f(t^+) = \lim_{u \to t^+} f(u) = \lim_{n \to \infty} f(t_n) \text{ for } t_n > t \rightarrow \infty \]

- **left-hand limit**
  \[ f(t^-) = \lim_{u \to t^-} f(u) = \lim_{n \to \infty} f(s_n) \text{ for } s_n < t \rightarrow \infty \]

exist and are real numbers for all $t \in (a, b)$.

**Proof.** This is analogous to the proof that a bounded monotone sequence converges. \(\square\)
Monotonic Functions

We say that $f$ has a *simple jump discontinuity* at $t$ if the one-sided limits $f(t^-)$ and $f(t^+)$ both exist but $f$ is not continuous at $t$.

Note that there are two ways $f$ can have a simple jump discontinuity at $t$: either $f(t^+) \neq f(t^-)$, or $f(t^+) = f(t^-) \neq f(t)$.

The previous theorem says that monotonic functions have only simple jump discontinuities. Note that monotonicity also implies that $f(t^-) \leq f(t) \leq f(t^+)$. So a monotonic function has a discontinuity at $t$ if and only if $f(t^+) \neq f(t^-)$. 
\[ f(x) = \frac{1}{x} \]

\[ f(x_1^+) = f(x_1^-) = f(x_1) \]

\[ f(x_2^+) = f(x_2^-) = f(x_2) \]

\[ U_1 = (-1, 1) \]
\[ U_2 = (-1, 2) \]
\[ U_3 = (-1, 3) \]
\[ U_4 = (-1, 4) \]

\[ x_1, x_2, x_3, \ldots \]
Monotonic Functions

A monotonic function is continuous “almost everywhere” — except for at most countably many points.

**Theorem 4** (Thm. 6.28). Let \( a, b \in \mathbb{R} \) with \( a < b \), and let \( f : (a, b) \to \mathbb{R} \) be monotonically increasing. Then

\[
D = \{ t \in (a, b) : f \text{ is discontinuous at } t \}
\]

is finite (possibly empty) or countable.

**Proof.** If \( t \in D \), then \( f(t^-) < f(t^+) \) (if the left- and right-hand limits agreed, then by monotonicity they would have to equal \( f(t) \), so \( f \) would be continuous at \( t \)). \( \mathbb{Q} \) is dense in \( \mathbb{R} \), that is, if
$x, y \in \mathbb{R}$ and $x < y$ then $\exists r \in \mathbb{Q}$ such that $x < r < y$. So for every $t \in D$ we may choose $r(t) \in \mathbb{Q}$ such that

$$f(t^-) < r(t) < f(t^+)$$

This defines a function $r : D \to \mathbb{Q}$. Notice that

$$s > t \Rightarrow f(s^-) \geq f(t^+)$$

so

$$s > t, s, t \in D \Rightarrow r(s) > f(s^-) \geq f(t^+) > r(t)$$

so $r(s) \neq r(t)$. Therefore, $r$ is one-to-one, so it is a bijection from $D$ to a subset of $\mathbb{Q}$. Thus $D$ is finite or countable. \hfill \Box
Cauchy Sequences and Complete Metric Spaces

Roughly, a metric space is complete if “every sequence that ought to converge to a limit has a limit to converge to.”

Recall that $x_n \to x$ means

$$\forall \varepsilon > 0 \ \exists N(\varepsilon/2) \text{ s.t. } n > N(\varepsilon/2) \Rightarrow d(x_n, x) < \frac{\varepsilon}{2}$$

Observe that if $n, m > N(\varepsilon/2)$, then

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
Cauchy Sequences and Complete Metric Spaces

This motivates the following definition:

**Definition 2.** A sequence \( \{x_n\} \) in a metric space \((X, d)\) is Cauchy if

\[
\forall \varepsilon > 0 \ \exists N(\varepsilon) \text{ s.t. } n, m > N(\varepsilon) \Rightarrow d(x_n, x_m) < \varepsilon
\]

A Cauchy sequence is trying really hard to converge, but there may not be anything for it to converge to.
Cauchy Sequences and Complete Metric Spaces

Any sequence that \textbf{does} converge must be Cauchy:

\textbf{Theorem 5} (Thm. 7.2). \textit{Every convergent sequence in a metric space is Cauchy.}

\textit{Proof.} We just did it: Let \(x_n \to x\). For every \(\varepsilon > 0\) \(\exists N\) such that \(n > N \Rightarrow d(x_n, x) < \varepsilon/2\). Then

\[ m, n > N \Rightarrow d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \]

\square
**Example:** Let $X = (0, 1]$ and $d$ be the Euclidean metric. Let $x_n = \frac{1}{n}$. Then $x_n \to 0$ in $\mathbb{E}^1$, so $\{x_n\}$ is Cauchy in $\mathbb{E}^1$. Thus $\{x_n\}$ is Cauchy in $(X, d)$. But $\{x_n\}$ does not converge in $(X, d)$.

The Cauchy property depends only on the sequence and the metric $d$, not on the ambient metric space:

$\{x_n\}$ is Cauchy in $(X, d)$, but $\{x_n\}$ does not **converge** in $(X, d)$ because the point it is trying to converge to (0) is not an element of $X$. 
Complete Metric Spaces and Banach Spaces

Where does every Cauchy sequence get what it wants?

**Definition 3.** A metric space \( (X, d) \) is complete if every Cauchy sequence \( \{x_n\} \subseteq X \) converges to a limit \( x \in X \).

**Definition 4.** A Banach space is a normed space that is complete in the metric generated by its norm.
Complete Metric Spaces and Banach Spaces

**Example:** Consider the earlier example of $X = (0, 1]$ with $d$ the usual Euclidean metric. The sequence $\{x_n\}$ with $x_n = \frac{1}{n}$ is Cauchy but does not converge, so $((0, 1], d)$ is not complete.

**Example:** $\mathbb{Q}$ is not complete in the Euclidean metric. To see this, let

$$x_n = \frac{\lfloor 10^n \sqrt{2} \rfloor}{10^n}$$

where $\lfloor y \rfloor$ is the greatest integer less than or equal to $y$; $x_n$ is just equal to the decimal expansion of $\sqrt{2}$ to $n$ digits past the decimal point. Clearly, $x_n$ is rational. $|x_n - \sqrt{2}| \leq 10^{-n}$, so $x_n \to \sqrt{2}$ in $\mathbb{E}^1$, so $\{x_n\}$ is Cauchy in $\mathbb{E}^1$, hence Cauchy in $\mathbb{Q}$; since $\sqrt{2} \notin \mathbb{Q}$, $\{x_n\}$ is not convergent in $\mathbb{Q}$, so $\mathbb{Q}$ is not complete.
Complete Metric Spaces and Banach Spaces

**Theorem 6** (Thm. 7.10). $\mathbb{R}$ is complete with the usual metric (so $\mathbb{E}^1$ is a Banach space).

**Proof.** Suppose $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}$. Fix $\varepsilon > 0$. Find $N(\varepsilon/2)$ such that

$$n, m > N(\varepsilon/2) \Rightarrow |x_n - x_m| < \frac{\varepsilon}{2}$$

Let

$$\alpha_n = \sup \{x_k : k \geq n\}$$

$$\beta_n = \inf \{x_k : k \geq n\}$$

Fix $m > N(\varepsilon/2)$. Then

$$k \geq m \Rightarrow k > N(\varepsilon/2) \Rightarrow x_k < x_m + \frac{\varepsilon}{2}$$

$$\Rightarrow \alpha_m = \sup \{x_k : k \geq m\} \leq x_m + \frac{\varepsilon}{2}$$
Claim: If 
\[ 0 \leq \limsup x_n - \liminf x_n \leq \varepsilon \]

\[ \forall \varepsilon > 0, \text{ then } \limsup x_n = \liminf x_n \]

Claim: If \( x_n \) and \( 0 \leq r \leq \varepsilon \)

\[ A \geq 0, \text{ then } r = 0 \]

Proof: Suppose not, so \( r > 0 \). Then \( r > 0 \).
Since \( \alpha_m < \infty \),

\[
\limsup x_n = \lim_{n \to \infty} \alpha_n \leq \alpha_m \leq x_m + \frac{\varepsilon}{2}
\]
since the sequence \( \{\alpha_n\} \) is decreasing. Similarly,

\[
\liminf x_n \geq x_m - \frac{\varepsilon}{2}
\]

Therefore,

\[
\frac{\varepsilon}{2} \leq \liminf x_n \leq \limsup x_n \leq \frac{\varepsilon}{2}
\]

\[
0 \leq \limsup x_n - \liminf x_n \leq \varepsilon
\]

Since \( \varepsilon \) is arbitrary,

\[
\limsup x_n = \liminf x_n \in \mathbb{R}
\]

Thus \( \lim_{n \to \infty} x_n \) exists and is real, so \( \{x_n\} \) is convergent. \( \square \)
Complete Metric Spaces and Banach Spaces

**Theorem 7** (Thm. 7.11). $\mathbb{E}^n$ is complete for every $n \in \mathbb{N}$.

*Proof.* See de la Fuente. \qed
Complete Metric Spaces and Banach Spaces

**Theorem 8** (Thm. 7.9). Suppose \((X, d)\) is a complete metric space and \(Y \subseteq X\). Then \((Y, d) = (Y, d|_Y)\) is complete if and only if \(Y\) is a closed subset of \(X\).

*Proof.* Suppose \((Y, d)\) is complete. We need to show that \(Y\) is closed. Consider a sequence \(\{y_n\} \subseteq Y\) such that \(y_n \to_{(X,d)} x \in X\). Then \(\{y_n\}\) is Cauchy in \(X\), hence Cauchy in \(Y\); since \(Y\) is complete, \(y_n \to_{(Y,d)} y\) for some \(y \in Y\). Therefore, \(y_n \to_{(X,d)} y\). By uniqueness of limits, \(y = x\), so \(x \in Y\). Thus \(Y\) is closed.

Conversely, suppose \(Y\) is closed. We need to show that \(Y\) is complete. Let \(\{y_n\}\) be a Cauchy sequence in \(Y\). Then \(\{y_n\}\) is Cauchy in \(X\), hence convergent, so \(y_n \to_{(X,d)} x\) for some \(x \in X\). Since \(Y\) is closed, \(x \in Y\), so \(y_n \to_{(Y,d)} x \in Y\). Thus \(Y\) is complete. \(\Box\)
Complete Metric Spaces and Banach Spaces

**Theorem 9** (Thm. 7.12). Given $X \subseteq \mathbb{R}^n$, let $C(X)$ be the set of bounded continuous functions from $X$ to $\mathbb{R}$ with

$$
\|f\|_{\infty} = \sup\{|f(x)| : x \in X\}
$$

Then $C(X)$ is a Banach space.
Contractions

**Definition 5.** Let \((X, d)\) be a nonempty complete metric space. An operator is a function \(T : X \rightarrow X\).

An operator \(T\) is a contraction of modulus \(\beta\) if \(0 \leq \beta < 1\) and

\[
d(T(x), T(y)) \leq \beta d(x, y) \quad \forall x, y \in X
\]

A contraction shrinks distances by a **uniform** factor \(\beta < 1\).
Contractions

**Theorem 10.** Every contraction is uniformly continuous.

*Proof.* Fix \( \varepsilon > 0 \). Let \( \delta = \frac{\varepsilon}{\beta} \). Then \( \forall x, y \) such that \( d(x, y) < \delta \),

\[
d(T(x), T(y)) \leq \beta d(x, y) < \beta \delta = \varepsilon
\]

Note that a contraction is Lipschitz continuous with Lipschitz constant \( \beta < 1 \) (and hence also uniformly continuous).
Contractions and Fixed Points

Definition 6. A fixed point of an operator $T$ is point $x^* \in X$ such that $T(x^*) = x^*$. 
? \( \exists x \) s.t. \( T(x) = x \)?

\[ x^* = T(x^*) \]

\( \text{graph } T \)

\( \{ (x, y) \in X \times X : x = y \} \)
Contraction Mapping Theorem

Theorem 11 (Thm. 7.16, Contraction Mapping Theorem). Let $(X, d)$ be a nonempty complete metric space and $T : X \to X$ a contraction with modulus $\beta < 1$. Then

1. $T$ has a unique fixed point $x^*$.

2. For every $x_0 \in X$, the sequence $\{x_n\}$ where

   $x_1 = T(x_0), x_2 = T(x_1) = T(T(x_0)), \ldots, x_n = T(x_{n-1})$ for each $n$

   converges to $x^*$. 
Note that the theorem asserts both the existence and uniqueness of the fixed point, as well as giving an algorithm to find the fixed point of a contraction.

Also note that the algorithm generates a sequence that converges to the fixed point for any initial point $x_0$.

Later in the course we will discuss more general fixed point theorems which, in contrast, only guarantee existence, and are not constructive.
Proof. Define the sequence \( \{x_n\} \) as above by first fixing \( x_0 \in X \) and then letting \( x_n = T(x_{n-1}) = T^n(x_0) \) for \( n = 1, 2, \ldots \), where \( T^n = T \circ T \circ \ldots \circ T \) is the \( n \)-fold iteration of \( T \). We first show that \( \{x_n\} \) is Cauchy, and hence converges to a limit \( x \). Then

\[
\begin{align*}
d(x_{n+1}, x_n) &= d(T(x_n), T(x_{n-1})) \\
&\leq \beta d(x_n, x_{n-1}) = \beta d(T(x_{n-1}), T(x_{n-2})) \\
&\leq \beta^2 d(x_{n-1}, x_{n-2}) \\
&\vdots \\
&\leq \beta^n d(x_1, x_0)
\end{align*}
\]
Then for any $n > m$,

$$
\begin{align*}
    d(x_n, x_m) & \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\
    & \leq (\beta^{n-1} + \beta^{n-2} + \cdots + \beta^m) d(x_1, x_0) \\
    & = d(x_1, x_0) \sum_{\ell=m}^{n-1} \beta^\ell \\
    & < d(x_1, x_0) \sum_{\ell=m}^{\infty} \beta^\ell \\
    & = \frac{\beta^m}{1 - \beta} d(x_1, x_0) \quad \text{(sum of a geometric series)}
\end{align*}
$$

Fix $\varepsilon > 0$. Since $\frac{\beta^m}{1 - \beta} \to 0$ as $m \to \infty$, choose $N(\varepsilon)$ such that for any $m > N(\varepsilon)$, $\frac{\beta^m}{1 - \beta} < \frac{\varepsilon}{d(x_1, x_0)}$. Then for $n, m > N(\varepsilon)$,

$$
    d(x_n, x_m) \leq \frac{\beta^m}{1 - \beta} d(x_1, x_0) < \varepsilon
$$
Therefore, \( \{x_n\} \) is Cauchy. Since \((X, d)\) is complete, \(x_n \rightarrow x^*\) for some \(x^* \in X\).

Next, we show that \(x^*\) is a fixed point of \(T\).

\[
T(x^*) = T \left( \lim_{n \rightarrow \infty} x_n \right) \\
= \lim_{n \rightarrow \infty} T(x_n) \text{ since } T \text{ is continuous} \\
= \lim_{n \rightarrow \infty} x_{n+1} \text{ (def. of } x_n) \\
= x^*
\]

so \(x^*\) is a fixed point of \(T\).

Finally, we show that there is at most one fixed point. Suppose \(x^*\) and \(y^*\) are both fixed points of \(T\), so \(T(x^*) = x^*\) and \(T(y^*) = y^*\).
Then

\[
d(x^*, y^*) = d(T(x^*), T(y^*))
\leq \beta d(x^*, y^*)
\Rightarrow (1 - \beta)d(x^*, y^*) \leq 0
\Rightarrow d(x^*, y^*) \leq 0
\]

So \(d(x^*, y^*) = 0\), which implies \(x^* = y^*\). \(\square\)
Continuous Dependence on Parameters

Theorem 12. (Thm. 7.18’, Continuous Dependence on Parameters) Let \((X,d)\) and \((\Omega,\rho)\) be two metric spaces and \(T : X \times \Omega \to X\). For each \(\omega \in \Omega\) let \(T_\omega : X \to X\) be defined by

\[ T_\omega(x) = T(x,\omega) \]

Suppose \((X,d)\) is complete, \(T\) is continuous in \(\omega\), that is \(T(x,\cdot) : \Omega \to X\) is continuous for each \(x \in X\), and \(\exists \beta < 1\) such that \(T_\omega\) is a contraction of modulus \(\beta\) \(\forall \omega \in \Omega\). Then the fixed point function \(x^* : \Omega \to X\) defined by

\[ x^*(\omega) = T_\omega(x^*(\omega)) \]

is continuous.
Blackwell’s Sufficient Conditions

An important result due to Blackwell gives a set of sufficient conditions for an operator to be a contraction that is particularly useful in dynamic programming problems.

Let $X$ be a set, and let $B(X)$ be the set of all bounded functions from $X$ to $\mathbb{R}$. Then $(B(X), \| \cdot \|_\infty)$ is a normed vector space.

Notice that below we use shorthand notation that identifies a constant function with its constant value in $\mathbb{R}$, that is, we write interchangeably $a \in \mathbb{R}$ and $a : X \to \mathbb{R}$ to denote the function such that $a(x) = a \ \forall x \in X$. 
Blackwell’s Sufficient Conditions

Theorem 13. (Blackwell’s Sufficient Conditions) Consider $B(X)$ with the sup norm $\| \cdot \|_\infty$. Let $T : B(X) \to B(X)$ be an operator satisfying

1. (monotonicity) $f(x) \leq g(x) \ \forall x \in X \Rightarrow (Tf)(x) \leq (Tg)(x) \ \forall x \in X$

2. (discounting) $\exists \beta \in (0, 1)$ such that for every $a \geq 0$ and $x \in X$,

   $$(T(f + a))(x) \leq (Tf)(x) + \beta a$$

Then $T$ is a contraction with modulus $\beta$. 
Proof. Fix \( f, g \in B(X) \). By the definition of the sup norm,
\[
f(x) \leq g(x) + \|f - g\|_\infty \quad \forall x \in X
\]
Then
\[
(Tf)(x) \leq (T(g + \|f - g\|_\infty))(x) \quad \forall x \in X \quad \text{(monotonicity)}
\]
\[
\leq (Tg)(x) + \beta\|f - g\|_\infty \quad \forall x \in X \quad \text{(discounting)}
\]
Thus
\[
(Tf)(x) - (Tg)(x) \leq \beta\|f - g\|_\infty \quad \forall x \in X
\]
Reversing the roles of \( f \) and \( g \) above gives
\[
(Tg)(x) - (Tf)(x) \leq \beta\|f - g\|_\infty \quad \forall x \in X
\]
Thus
\[
\|T(f) - T(g)\|_\infty \leq \beta\|f - g\|_\infty
\]
Thus \( T \) is a contraction with modulus \( \beta \)