Econ 204 2016

Lecture 6

Outline

1. Open Covers
2. Compactness
3. Sequential Compactness
4. Totally Bounded Sets
5. Heine-Borel Theorem
6. Extreme Value Theorem
Open Covers

**Definition 1.** A *collection of sets* $\mathcal{U} = \{U_{\lambda} : \lambda \in \Lambda\}$ in a metric space $(X, d)$ is an open cover of $A$ if $U_{\lambda}$ is open for all $\lambda \in \Lambda$ and

$$\bigcup_{\lambda \in \Lambda} U_{\lambda} \supseteq A$$

Notice that $\Lambda$ may be finite, countably infinite, or uncountable.
Compactness

Definition 2. A set $A$ in a metric space is compact if every open cover of $A$ contains a finite subcover of $A$. In other words, if \( \{U_\lambda : \lambda \in \Lambda\} \) is an open cover of $A$, there exist $n \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_n \in \Lambda$ such that

$$A \subseteq U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}$$

This definition does not say “$A$ has a finite open cover” (fortunately, since this is vacuous...).

Instead for any arbitrary open cover you must specify a finite subcover of this given open cover.
Compactness

**Example:** $(0, 1]$ is not compact in $\mathbb{E}^1$.

To see this, let

$$\mathcal{U} = \left\{ U_m = \left( \frac{1}{m}, 2 \right) : m \in \mathbb{N} \right\}$$

Then

$$\bigcup_{m \in \mathbb{N}} U_m = (0, 2) \supset (0, 1]$$
Given any finite subset \( \{U_{m_1}, \ldots, U_{m_n}\} \) of \( \mathcal{U} \), let

\[
m = \max\{m_1, \ldots, m_n\}
\]

Then

\[
\bigcup_{i=1}^{n} U_{m_i} = U_m = \left( \frac{1}{m}, 2 \right) \not\supset (0, 1]
\]

So \((0, 1]\) is not compact.

What about \([0, 1]\)? This argument doesn’t work...
Compactness

**Example:** $[0, \infty)$ is closed but not compact.

To see that $[0, \infty)$ is not compact, let

$$\mathcal{U} = \{U_m = (-1, m) : m \in \mathbb{N}\}$$

Given any finite subset

$$\{U_{m_1}, \ldots, U_{m_n}\}$$

of $\mathcal{U}$, let

$$m = \max\{m_1, \ldots, m_n\}$$

Then

$$U_{m_1} \cup \cdots \cup U_{m_n} = (-1, m) \not\supseteq [0, \infty)$$
\( f(a), f(b) \)

\( U_1 = (-1,1) \)
\( U_2 = (-1,2) \)
\( U_3 = (-1,3) \)
Compactness

Theorem 1 (Thm. 8.14). Every closed subset $A$ of a compact metric space $(X, d)$ is compact.

Proof. Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover of $A$. In order to use the compactness of $X$, we need to produce an open cover of $X$. There are two ways to do this:

\[
U'_\lambda = U_\lambda \cup (X \setminus A) \\
\Lambda' = \Lambda \cup \{\lambda_0\}, \quad U_{\lambda_0} = X \setminus A
\]

We choose the first path, and let

\[
U'_\lambda = U_\lambda \cup (X \setminus A)
\]
$U_\lambda' = U_\lambda \cup (X \setminus A)$
Since $A$ is closed, $X \setminus A$ is open; since $U_\lambda$ is open, so is $U'_\lambda$.

Then $x \in X \Rightarrow x \in A$ or $x \in X \setminus A$. If $x \in A$, $\exists \lambda \in \Lambda$ s.t. $x \in U_\lambda \subseteq U'_\lambda$. If instead $x \in X \setminus A$, then $\forall \lambda \in \Lambda$, $x \in U'_\lambda$. Therefore, $X \subseteq \bigcup_{\lambda \in \Lambda} U'_\lambda$, so $\{U'_\lambda : \lambda \in \Lambda\}$ is an open cover of $X$.

Since $X$ is compact,

$$\exists \lambda_1, \ldots, \lambda_n \in \Lambda \text{ s.t. } X \subseteq U'_\lambda_1 \cup \cdots \cup U'_\lambda_n$$

Then

$$a \in A \Rightarrow a \in X$$
$$\Rightarrow a \in U'_\lambda_i \text{ for some } i$$
$$\Rightarrow a \in U_\lambda_i \cup (X \setminus A)$$
$$\Rightarrow a \in U_\lambda_i$$
so

\[ A \subseteq U_{\lambda_1} \cup \cdots \cup U_{\lambda_n} \]

Thus \( A \) is compact. \( \square \)
Compactness

closed $\not\Rightarrow$ compact, but the converse is true:

**Theorem 2** (Thm. 8.15). *If $A$ is a compact subset of the metric space $(X, d)$, then $A$ is closed.*

*Proof.* Suppose by way of contradiction that $A$ is not closed. Then $X \setminus A$ is not open, so we can find a point $x \in X \setminus A$ such that, for every $\varepsilon > 0$, $A \cap B_\varepsilon(x) \neq \emptyset$, and hence $A \cap B_\varepsilon[x] \neq \emptyset$. For $n \in \mathbb{N}$, let

$$U_n = X \setminus B_\varepsilon^n[x]$$
$U_n = X \setminus B_{1/n}(x)$
Each $U_n$ is open, and

$$\bigcup_{n \in \mathbb{N}} U_n = X \setminus \{x\} \supseteq A$$

since $x \notin A$. Therefore, $\{U_n : n \in \mathbb{N}\}$ is an open cover for $A$. Since $A$ is compact, there is a finite subcover $\{U_{n_1}, \ldots, U_{n_k}\}$. Let $n = \max\{n_1, \ldots, n_k\}$. Then

$$U_n = X \setminus B_1^n[x]$$

$$\supseteq X \setminus B_1^n[x] (j = 1, \ldots, k)$$

$$U_n \supseteq \bigcup_{j=1}^k U_{n_j}$$

$$\supseteq A$$

But $A \cap B_1^n[x] \neq \emptyset$, so $A \not\subseteq X \setminus B_1^n[x] = U_n$, a contradiction which proves that $A$ is closed. \qed
Sequential Compactness

**Definition 3.** A set $A$ in a metric space $(X, d)$ is sequentially compact if every sequence of elements of $A$ contains a convergent subsequence whose limit lies in $A$. 
Sequential Compactness

**Theorem 3** (Thms. 8.5, 8.11). A set \( A \) in a metric space \((X, d)\) is compact if and only if it is sequentially compact.

**Proof.** Suppose \( A \) is compact. We will show that \( A \) is sequentially compact.

If not, we can find a sequence \( \{x_n\} \) of elements of \( A \) such that no subsequence converges to any element of \( A \). Recall that \( a \) is a cluster point of the sequence \( \{x_n\} \) means that

\[
\forall \varepsilon > 0 \quad \{n : x_n \in B_{\varepsilon}(a)\} \text{ is infinite}
\]

and this is equivalent to the statement that there is a subsequence \( \{x_{n_k}\} \) converging to \( a \). Thus, no element \( a \in A \) can be a cluster point for \( \{x_n\} \), and hence

\[
\forall a \in A \quad \exists \varepsilon_a > 0 \text{ s.t. } \{n : x_n \in B_{\varepsilon_a}(a)\} \text{ is finite} \quad (1)
\]
Then
\[ \{ B_{\varepsilon_a}(a) : a \in A \} \]
is an open cover of \( A \) (if \( A \) is uncountable, it will be an uncountable open cover). Since \( A \) is compact, there is a finite subcover
\[ \{ B_{\varepsilon_{a_1}}(a_1), \ldots, B_{\varepsilon_{a_m}}(a_m) \} \]
Then
\[
N = \{ n : x_n \in A \} \\
\subseteq \{ n : x_n \in (B_{\varepsilon_{a_1}}(a_1) \cup \cdots \cup B_{\varepsilon_{a_m}}(a_m)) \} \\
= \{ n : x_n \in B_{\varepsilon_{a_1}}(a_1) \} \cup \cdots \cup \{ n : x_n \in B_{\varepsilon_{a_m}}(a_m) \}
\]
so \( N \) is contained in a finite union of sets, each of which is finite by Equation (1). Thus, \( N \) must be finite, a contradiction which proves that \( A \) is sequentially compact.
For the converse, see de la Fuente.
Totally Bounded Sets

Definition 4. A set $A$ in a metric space $(X, d)$ is totally bounded if, for every $\varepsilon > 0$,

$$\exists x_1, \ldots, x_n \in A \text{ s.t. } A \subseteq \bigcup_{i=1}^{n} B_{\varepsilon}(x_i)$$
Totally Bounded Sets

**Example:** Take $A = [0,1]$ with the Euclidean metric. Given $\varepsilon > 0$, let $n > \frac{1}{\varepsilon}$. Then we may take

$$x_1 = \frac{1}{n}, x_2 = \frac{2}{n}, \ldots, x_{n-1} = \frac{n-1}{n}$$

Then $[0,1] \subset \bigcup_{k=1}^{n-1} B_\varepsilon(\frac{k}{n})$. 
Totally Bounded Sets

**Example:** Consider $X = [0,1]$ with the discrete metric

$$d(x, y) = \begin{cases} 
1 & \text{if } x \neq y \\
0 & \text{if } x = y
\end{cases}$$

$X$ is not totally bounded. To see this, take $\varepsilon = \frac{1}{2}$. Then for any $x$, $B_\varepsilon(x) = \{x\}$, so given any finite set $x_1, \ldots, x_n$,

$$\bigcup_{i=1}^{n} B_\varepsilon(x_i) = \{x_1, \ldots, x_n\} \not\subseteq [0,1]$$

However, $X$ is bounded because $X = B_2(0)$. 
Totally Bounded Sets

Note that any totally bounded set in a metric space \((X, d)\) is also bounded. To see this, let \(A \subset X\) be totally bounded. Then \(\exists x_1, \ldots, x_n \in A\) such that \(A \subset B_1(x_1) \cup \cdots \cup B_1(x_n)\). Let

\[
M = 1 + d(x_1, x_2) + \cdots + d(x_{n-1}, x_n)
\]

Then \(M < \infty\). Now fix \(a \in A\). We claim \(d(a, x_1) < M\). To see this, notice that there is some \(n_a \in \{1, \ldots, n\}\) for which \(a \in B_1(x_{n_a})\). Then

\[
d(a, x_1) \leq d(a, x_{n_a}) + \sum_{k=1}^{n} d(x_k, x_{k+1})
\]

\[
< 1 + \sum_{k=1}^{n} d(x_k, x_{k+1})
\]

\[
= M
\]
Totally Bounded Sets

**Remark 4.** Every compact subset of a metric space is totally bounded:

Fix $\varepsilon$ and consider the open cover

$$\mathcal{U}_\varepsilon = \{B_\varepsilon(a) : a \in A\}$$

If $A$ is compact, then every open cover of $A$ has a finite subcover; in particular, $\mathcal{U}_\varepsilon$ must have a finite subcover, but this just says that $A$ is totally bounded.
Compactness and Totally Bounded Sets

**Theorem 5** (Thm. 8.16). *Let $A$ be a subset of a metric space $(X, d)$. Then $A$ is compact if and only if it is complete and totally bounded.*

*Proof.* Here is a sketch of the proof; see de la Fuente for details. Compact implies totally bounded (Remark 4). Suppose $\{x_n\}$ is a Cauchy sequence in $A$. Since $A$ is compact, $A$ is sequentially compact, hence $\{x_n\}$ has a convergent subsequence $x_{n_k} \to a \in A$. Since $\{x_n\}$ is Cauchy, $x_n \to a$ (why?), so $A$ is complete.

Conversely, suppose $A$ is complete and totally bounded. Let $\{x_n\}$ be a sequence in $A$. Because $A$ is totally bounded, we can extract a Cauchy subsequence $\{x_{n_k}\}$ (why?). Because $A$ is complete, $x_{n_k} \to a$ for some $a \in A$, which shows that $A$ is sequentially compact and hence compact. \(\square\)
Compact $\iff$ Closed and Totally Bounded

Putting these together:

**Corollary 1.** Let $A$ be a subset of a complete metric space $(X, d)$. Then $A$ is compact if and only if $A$ is closed and totally bounded.

\[
\begin{align*}
A \text{ compact} & \implies A \text{ complete and totally bounded} \\
& \implies A \text{ closed and totally bounded} \\
A \text{ closed and totally bounded} & \implies A \text{ complete and totally bounded} \\
& \implies A \text{ compact}
\end{align*}
\]
Example: [0, 1] is compact in $E^1$.

Note: compact $\Rightarrow$ closed and bounded, but converse need not be true.

E.g. [0, 1] with the discrete metric.
Heine-Borel Theorem - $\mathbb{E}^1$

**Theorem 6** (Thm. 8.19, Heine-Borel). *If $A \subseteq \mathbb{E}^1$, then $A$ is compact if and only if $A$ is closed and bounded.*

*Proof.* Let $A$ be a closed, bounded subset of $\mathbb{R}$. Then $A \subseteq [a, b]$ for some interval $[a, b]$. Let $\{x_n\}$ be a sequence of elements of $[a, b]$. By the Bolzano-Weierstrass Theorem, $\{x_n\}$ contains a convergent subsequence with limit $x \in \mathbb{R}$. Since $[a, b]$ is closed, $x \in [a, b]$. Thus, we have shown that $[a, b]$ is sequentially compact, hence compact. $A$ is a closed subset of $[a, b]$, hence $A$ is compact.

Conversely, if $A$ is compact, $A$ is closed and bounded. $\square$
Heine-Borel Theorem - $\mathbb{E}^n$

**Theorem 7** (Thm. 8.20, Heine-Borel). If $A \subseteq \mathbb{E}^n$, then $A$ is compact if and only if $A$ is closed and bounded.

*Proof.* See de la Fuente. \hfill \square

**Example:** The closed interval

$$[a, b] = \{x \in \mathbb{R}^n : a_i \leq x_i \leq b_i \text{ for each } i = 1, \ldots, n\}$$

is compact in $\mathbb{E}^n$ for any $a, b \in \mathbb{R}^n$. 

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Continuous Images of Compact Sets

**Theorem 8 (8.21).** Let \((X, d)\) and \((Y, \rho)\) be metric spaces. If \(f : X \to Y\) is continuous and \(C\) is a compact subset of \((X, d)\), then \(f(C)\) is compact in \((Y, \rho)\).

**Proof.** There is a proof in de la Fuente using sequential compactness. Here we give an alternative proof using directly the open cover definition of compactness.

Let \(\{U_\lambda : \lambda \in \Lambda\}\) be an open cover of \(f(C)\). For each point \(c \in C\), \(f(c) \in f(C)\) so \(f(c) \in U_{\lambda_c}\) for some \(\lambda_c \in \Lambda\), that is, \(c \in f^{-1}(U_{\lambda_c})\). Thus the collection \(\{f^{-1}(U_\lambda) : \lambda \in \Lambda\}\) is a cover of \(C\); in addition, since \(f\) is continuous, each set \(f^{-1}(U_\lambda)\) is
open in $C$, so $\{f^{-1}(U_\lambda) : \lambda \in \Lambda\}$ is an open cover of $C$. Since $C$ is compact, there is a finite subcover

$$\{f^{-1}(U_{\lambda_1}), \ldots, f^{-1}(U_{\lambda_n})\}$$

of $C$. Given $x \in f(C)$, there exists $c \in C$ such that $f(c) = x$, and $c \in f^{-1}(U_{\lambda_i})$ for some $i$, so $x \in U_{\lambda_i}$. Thus, $\{U_{\lambda_1}, \ldots, U_{\lambda_n}\}$ is a finite subcover of $f(C)$, so $f(C)$ is compact. \qed
Extreme Value Theorem

**Corollary 2** (Thm. 8.22, Extreme Value Theorem). *Let $C$ be a compact set in a metric space $(X, d)$, and suppose $f : C \to \mathbb{R}$ is continuous. Then $f$ is bounded on $C$ and attains its minimum and maximum on $C$.*

*Proof.* $f(C)$ is compact by Theorem 8.21, hence closed and bounded. Let $M = \sup f(C)$; $M < \infty$. Then $\forall m > 0$ there exists $y_m \in f(C)$ such that

$$M - \frac{1}{m} \leq y_m \leq M$$

So $y_m \to M$ and $\{y_m\} \subseteq f(C)$. Since $f(C)$ is closed, $M \in f(C)$, i.e. there exists $c \in C$ such that $f(c) = M = \sup f(C)$, so $f$ attains its maximum at $c$. The proof for the minimum is similar. \qed
Compactness and Uniform Continuity

**Theorem 9** (Thm. 8.24). Let \((X, d)\) and \((Y, \rho)\) be metric spaces, \(C\) a compact subset of \(X\), and \(f : C \to Y\) continuous. Then \(f\) is uniformly continuous on \(C\).

**Proof.** Fix \(\varepsilon > 0\). We ignore \(X\) and consider \(f\) as defined on the metric space \((C, d)\). Given \(c \in C\), find \(\delta(c) > 0\) such that

\[
x \in C, \quad d(x, c) < 2\delta(c) \Rightarrow \rho(f(x), f(c)) < \frac{\varepsilon}{2}
\]

Let

\[
U_c = B_{\delta(c)}(c)
\]

Then

\[
\{U_c : c \in C\}
\]
is an open cover of $C$. Since $C$ is compact, there is a finite subcover

$$\{U_{c_1}, \ldots, U_{c_n}\}$$

Let

$$\delta = \min\{\delta(c_1), \ldots, \delta(c_n)\}$$

Given $x, y \in C$ with $d(x, y) < \delta$, note that $x \in U_{c_i}$ for some $i \in \{1, \ldots, n\}$, so $d(x, c_i) < \delta(c_i)$.

$$d(y, c_i) \leq d(y, x) + d(x, c_i)$$
$$< \delta + \delta(c_i)$$
$$\leq \delta(c_i) + \delta(c_i)$$
$$= 2\delta(c_i)$$
so

\[ \rho(f(x), f(y)) \leq \rho(f(x), f(c_i)) + \rho(f(c_i), f(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \]

which proves that \( f \) is uniformly continuous. \( \square \)