Definition 1 Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an open interval. $f$ is differentiable at $x \in I$ if
\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = a
\]
for some $a \in \mathbb{R}$.

This is equivalent to $\exists a \in \mathbb{R}$ such that:
\[
\lim_{h \to 0} \frac{f(x + h) - (f(x) + ah)}{h} = 0
\]
\[
\Leftrightarrow \forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } 0 < |h| < \delta \Rightarrow \left| \frac{f(x + h) - (f(x) + ah)}{h} \right| < \varepsilon
\]
\[
\Leftrightarrow \forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } 0 < |h| < \delta \Rightarrow \left| \frac{f(x + h) - (f(x) + ah)}{|h|} \right| < \varepsilon
\]
\[
\Leftrightarrow \lim_{h \to 0} \frac{|f(x + h) - (f(x) + ah)|}{|h|} = 0
\]

Recall that the limit considers $h$ near zero, but not $h = 0$.

Definition 2 If $X \subseteq \mathbb{R}^n$ is open, $f : X \to \mathbb{R}^m$ is differentiable at $x \in X$ if
\[
\exists T_x \in L(\mathbb{R}^n, \mathbb{R}^m) \text{ s.t. } \lim_{h \to 0, h \neq 0} \frac{|f(x + h) - (f(x) + T_x(h))|}{|h|} = 0 \tag{1}
\]
f is differentiable if it is differentiable at all $x \in X$.

Note that $T_x$ is uniquely determined by Equation (1). $h$ is a small, nonzero element of $\mathbb{R}^n$; $h \to 0$ from any direction, from above, below, along a spiral, etc. The definition requires that one linear operator $T_x$ works no matter how $h$ approaches zero. In this case, $f(x) + T_x(h)$ is the best linear approximation to $f(x + h)$ for small $h$.

Notation:

- $y = O(|h|^n)$ as $h \to 0$ – read “$y$ is big-Oh of $|h|^n$” – means
  \[
  \exists K, \delta > 0 \text{ s.t. } |h| < \delta \Rightarrow |y| \leq K|h|^n
  \]

\[\text{Recall } | \cdot | \text{ denotes the Euclidean distance.}\]
• \( y = o(|h|^n) \) as \( h \to 0 \) – read “\( y \) is little-oh of \(|h|^n\)” – means

\[
\lim_{h \to 0} \frac{|y|}{|h|^n} = 0
\]

Note that the statement \( y = O(|h|^{n+1}) \) as \( h \to 0 \) implies \( y = o(|h|^n) \) as \( h \to 0 \).

Also note that if \( y \) is either \( O(|h|^n) \) or \( o(|h|^n) \), then \( y \to 0 \) as \( h \to 0 \); the difference in whether \( y \) is “big-Oh” or “little-oh” tells us something about the rate at which \( y \to 0 \).

Using this notation, note that \( f \) is differentiable at \( x \) \( \iff \exists T_x \in L(\mathbb{R}^n, \mathbb{R}^m) \) such that

\[
f(x + h) = f(x) + T_x(h) + o(h) \text{ as } h \to 0
\]

Notation:

• \( df_x \) is the linear transformation \( T_x \)
• \( Df(x) \) is the matrix of \( df_x \) with respect to the standard basis.
  This is called the Jacobian or Jacobian matrix of \( f \) at \( x \)
• \( E_f(h) = f(x + h) - (f(x) + df_x(h)) \) is the error term

Using this notation,

\[
f \text{ is differentiable at } x \iff E_f(h) = o(h) \text{ as } h \to 0
\]

Now compute \( Df(x) = (a_{ij}) \). Let \( \{e_1, \ldots, e_n\} \) be the standard basis of \( \mathbb{R}^n \). Look in direction \( e_j \) (note that \(|\gamma e_j| = |\gamma|\)).

\[
o(\gamma) = f(x + \gamma e_j) - (f(x) + T_x(\gamma e_j))
\]

\[
= f(x + \gamma e_j) - \left( f(x) + \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right)
\]

\[
= f(x + \gamma e_j) - \left( f(x) + \begin{pmatrix} \gamma a_{1j} \\ \vdots \\ \gamma a_{mj} \end{pmatrix} \right)
\]
For $i = 1, \ldots, m$, let $f^i$ denote the $i^{th}$ component of the function $f$:

$$f^i(x + \gamma e_j) - (f^i(x) + \gamma a_{ij}) = o(\gamma)$$

so $a_{ij} = \frac{\partial f^i}{\partial x_j}(x)$

**Theorem 3 (Thm. 3.3)** Suppose $X \subseteq \mathbb{R}^n$ is open and $f : X \to \mathbb{R}^m$ is differentiable at $x \in X$. Then $\frac{\partial f^i}{\partial x_j}$ exists at $x$ for $1 \leq i \leq m$, $1 \leq j \leq n$, and

$$Df(x) = \begin{pmatrix}
\frac{\partial f^1}{\partial x_1}(x) & \cdots & \frac{\partial f^1}{\partial x_n}(x) \\
\vdots & \ddots & \vdots \\
\frac{\partial f^m}{\partial x_1}(x) & \cdots & \frac{\partial f^m}{\partial x_n}(x)
\end{pmatrix}$$

i.e. the Jacobian at $x$ is the matrix of partial derivatives at $x$.

**Remark:** If $f$ is differentiable at $x$, then all first-order partial derivatives $\frac{\partial f^i}{\partial x_j}$ exist at $x$. However, the converse is false: existence of all the first-order partial derivatives does not imply that $f$ is differentiable. The missing piece is continuity of the partial derivatives:

**Theorem 4 (Thm. 3.4)** If all the first-order partial derivatives $\frac{\partial f^i}{\partial x_j}$ ($1 \leq i \leq m$, $1 \leq j \leq n$) exist and are continuous at $x$, then $f$ is differentiable at $x$.

**Directional Derivatives:**

Suppose $X \subseteq \mathbb{R}^n$ open, $f : X \to \mathbb{R}^m$ is differentiable at $x$, and $|u| = 1$.

$$f(x + \gamma u) - (f(x) + T_x(\gamma u)) = o(\gamma) \text{ as } \gamma \to 0$$

$$\Rightarrow f(x + \gamma u) - (f(x) + \gamma T_x(u)) = o(\gamma) \text{ as } \gamma \to 0$$

$$\Rightarrow \lim_{\gamma \to 0} \frac{f(x + \gamma u) - f(x)}{\gamma} = T_x(u) = Df(x)u$$

i.e. the directional derivative in the direction $u$ (with $|u| = 1$) is

$$Df(x)u \in \mathbb{R}^m$$

**Theorem 5 (Thm. 3.5, Chain Rule)** Let $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$ be open, $f : X \to Y$, $g : Y \to \mathbb{R}^p$. Let $x_0 \in X$ and $F = g \circ f$. If $f$ is differentiable at $x_0$ and $g$ is differentiable at $f(x_0)$, then $F = g \circ f$ is differentiable at $x_0$ and

$$dF_{x_0} = dg_{f(x_0)} \circ df_{x_0}$$

(composition of linear transformations)

$$DF(x_0) = Dg(f(x_0))DF(x_0)$$

(matrix multiplication)
Remark: The statement is exactly the same as in the univariate case, except we replace the univariate derivative by a linear transformation. The proof is more or less the same, with a bit of linear algebra added.

Theorem 6 (Thm. 1.7, Mean Value Theorem, Univariate Case) Let \( a, b \in \mathbb{R} \). Suppose \( f : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\) and differentiable on \((a, b)\). Then there exists \( c \in (a, b) \) such that
\[
\frac{f(b) - f(a)}{b - a} = f'(c)
\]
that is, such that
\[
f(b) - f(a) = f'(c)(b - a)
\]

Proof: Consider the function
\[
g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)
\]
Then \( g(a) = 0 = g(b) \). See Figure 1. Note that for \( x \in (a, b) \),
\[
g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}
\]
so it suffices to find \( c \in (a, b) \) such that \( g'(c) = 0 \).

Case I: If \( g(x) = 0 \) for all \( x \in [a, b] \), choose an arbitrary \( c \in (a, b) \), and note that \( g'(c) = 0 \), so we are done.

Case II: Suppose \( g(x) > 0 \) for some \( x \in [a, b] \). Since \( g \) is continuous on \([a, b]\), it attains its maximum at some point \( c \in (a, b) \). Since \( g \) is differentiable at \( c \) and \( c \) is an interior point of the domain of \( g \), we have \( g'(c) = 0 \), and we are done.

Case III: If \( g(x) < 0 \) for some \( x \in [a, b] \), the argument is similar to that in Case II. ■

Remark: The Mean Value Theorem is useful for estimating bounds on functions and error terms in approximation of functions.

Notation:
\[
\ell(x, y) = \{ \alpha x + (1 - \alpha)y : \alpha \in [0, 1]\}
\]
is the line segment from \( x \) to \( y \).

Theorem 7 (Mean Value Theorem) Suppose \( f : \mathbb{R}^n \to \mathbb{R} \) is differentiable on an open set \( X \subseteq \mathbb{R}^n \), \( x, y \in X \) and \( \ell(x, y) \subseteq X \). Then there exists \( z \in \ell(x, y) \) such that
\[
f(y) - f(x) = Df(z)(y - x)
\]
Remark: This statement is different from Theorem 3.7 in de la Fuente. Notice that the statement is exactly the same as in the univariate case. For $f : \mathbb{R}^n \to \mathbb{R}^m$, we can apply the Mean Value Theorem to each component, to obtain $z_1, \ldots, z_m \in \ell(x, y)$ such that

$$f^i(y) - f^i(x) = Df^i(z_i)(y - x)$$

However, we cannot find a single $z$ which works for every component. Note that each $z_i \in \ell(x, y) \subset \mathbb{R}^n$; there are $m$ of them, one for each component in the range.

The following result plays the same role in estimating function values and error terms for functions taking values in $\mathbb{R}^m$ as the Mean Value Theorem plays for functions from $\mathbb{R}$ to $\mathbb{R}$.

**Theorem 8** Suppose $X \subset \mathbb{R}^n$ is open and $f : X \to \mathbb{R}^m$ is differentiable. If $x, y \in X$ and $\ell(x, y) \subseteq X$, then there exists $z \in \ell(x, y)$ such that

$$|f(y) - f(x)| \leq |df_z(y - x)|$$

$$\leq \|df_z\||y - x|$$

**Remark:** To understand why we don’t get equality, consider $f : [0, 1] \to \mathbb{R}^2$ defined by

$$f(t) = (\cos 2\pi t, \sin 2\pi t)$$

$f$ maps $[0, 1]$ to the unit circle in $\mathbb{R}^2$. Note that $f(0) = f(1) = (1, 0)$, so $|f(1) - f(0)| = 0$. However, for any $z \in [0, 1],$

$$|df_z(1 - 0)| = |2\pi(-\sin 2\pi z, \cos 2\pi z)|$$

$$= 2\pi \sqrt{\sin^2 2\pi z + \cos^2 2\pi z}$$

$$= 2\pi$$

**Section 4.4. Taylor’s Theorem**

**Theorem 9 (Thm. 1.9, Taylor’s Theorem in $\mathbb{R}^1$)** Let $f : I \to \mathbb{R}$ be $n$-times differentiable, where $I \subseteq \mathbb{R}$ is an open interval. If $x, x + h \in I$, then

$$f(x + h) = f(x) + \sum_{k=1}^{n-1} \frac{f^{(k)}(x)h^k}{k!} + E_n$$

where $f^{(k)}$ is the $k^{th}$ derivative of $f$ and

$$E_n = \frac{f^{(n)}(x + \lambda h)h^n}{n!} \text{ for some } \lambda \in (0, 1)$$
Motivation: Let

\[ T_n(h) = f(x) + \sum_{k=1}^{n} \frac{f^{(k)}(x)h^k}{k!} \]

\[ = f(x) + f'(x)h + \frac{f''(x)h^2}{2} + \cdots + \frac{f^{(n)}(x)h^n}{n!} \]

\[ T_n(0) = f(x) \]

\[ T'_n(h) = f'(x) + f''(x)h + \cdots + \frac{f^{(n)}(x)h^{n-1}}{(n-1)!} \]

\[ T'_n(0) = f'(x) \]

\[ T''_n(h) = f''(x) + \cdots + \frac{f^{(n)}(x)h^{n-2}}{(n-2)!} \]

\[ T''_n(0) = f''(x) \]

\[ \vdots \]

\[ T^{(n)}_n(0) = f^{(n)}(x) \]

so \( T_n(h) \) is the unique \( n^{th} \) degree polynomial such that

\[ T_n(0) = f(x) \]

\[ T'_n(0) = f'(x) \]

\[ \vdots \]

\[ T^{(n)}_n(0) = f^{(n)}(x) \]

The proof of the formula for the remainder \( E_n \) is essentially the Mean Value Theorem; the problem in applying it is that the point \( x + \lambda h \) is not known in advance.

Theorem 10 (Alternate Taylor’s Theorem in \( \mathbb{R}^1 \)) Let \( f : I \rightarrow \mathbb{R} \) be \( n \) times differentiable, where \( I \subseteq \mathbb{R} \) is an open interval and \( x \in I \). Then

\[ f(x + h) = f(x) + \sum_{k=1}^{n} \frac{f^{(k)}(x)h^k}{k!} + o(h^n) \text{ as } h \rightarrow 0 \]

If \( f \) is \((n + 1)\) times continuously differentiable (i.e. all derivatives up to order \( n + 1 \) exist and are continuous), then

\[ f(x + h) = f(x) + \sum_{k=1}^{n} \frac{f^{(k)}(x)h^k}{k!} + O\left(h^{n+1}\right) \text{ as } h \rightarrow 0 \]

Remark: The first equation in the statement of the theorem is essentially a restatement of the definition of the \( n^{th} \) derivative. The second statement is proven from Theorem 1.9, and the continuity of the derivative, hence the boundedness of the derivative on a neighborhood of \( x \).
Definition 11 Let $X \subseteq \mathbb{R}^n$ be open. A function $f : X \rightarrow \mathbb{R}^m$ is continuously differentiable on $X$ if

- $f$ is differentiable on $X$ and
- $df_x$ is a continuous function of $x$ from $X$ to $L(\mathbb{R}^n, \mathbb{R}^m)$, with operator norm $\|df_x\|

$f$ is $C^k$ if all partial derivatives of order less than or equal to $k$ exist and are continuous in $X$.

Theorem 12 (Thm. 4.3) Suppose $X \subseteq \mathbb{R}^n$ is open and $f : X \rightarrow \mathbb{R}^m$. Then $f$ is continuously differentiable on $X$ if and only if $f$ is $C^1$.

Remark: The notation in Taylor’s Theorem is difficult. If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, the quadratic terms are not hard for $m = 1$; for $m > 1$, we handle each component separately. For cubic and higher order terms, the notation is a mess.

Linear Terms:

Theorem 13 Suppose $X \subseteq \mathbb{R}^n$ is open and $x \in X$. If $f : X \rightarrow \mathbb{R}^m$ is differentiable, then

$$f(x + h) = f(x) + Df(x)h + o(h) \text{ as } h \rightarrow 0$$

The previous theorem is essentially a restatement of the definition of differentiability.

Theorem 14 (Corollary of 4.4) Suppose $X \subseteq \mathbb{R}^n$ is open and $x \in X$. If $f : X \rightarrow \mathbb{R}^m$ is $C^2$, then

$$f(x + h) = f(x) + Df(x)h + O(|h|^2) \text{ as } h \rightarrow 0$$

Quadratic Terms:

We treat each component of the function separately, so consider $f : X \rightarrow \mathbb{R}, X \subseteq \mathbb{R}^n$ an open set. Let

$$D^2f(x) = \begin{pmatrix}
\frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) \\
\frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(x) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 f}{\partial x_n \partial x_1}(x) & \frac{\partial^2 f}{\partial x_n \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(x)
\end{pmatrix}$$

$f \in C^2 \Rightarrow \frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x)$

$\Rightarrow D^2f(x)$ is symmetric

$\Rightarrow D^2f(x)$ has an orthonormal basis of eigenvectors

and thus can be diagonalized
**Theorem 15 (Stronger Version of Thm. 4.4)** Let $X \subseteq \mathbb{R}^n$ be open, $f : X \rightarrow \mathbb{R}$, $f \in C^2(X)$, and $x \in X$. Then

$$f(x + h) = f(x) + Df(x)h + \frac{1}{2} h^\top (D^2f(x))h + o\left(|h|^2\right) \text{ as } h \rightarrow 0$$

If $f \in C^3$,

$$f(x + h) = f(x) + Df(x)h + \frac{1}{2} h^\top (D^2f(x))h + O\left(|h|^3\right) \text{ as } h \rightarrow 0$$

**Remark:** de la Fuente assumes $X$ is convex. $X$ is said to be convex if, for every $x, y \in X$ and $\alpha \in [0, 1]$, $\alpha x + (1 - \alpha)y \in X$. Notice we don’t need this. Since $X$ is open,

$$x \in X \Rightarrow \exists \delta > 0 \text{ s.t. } B_\delta(x) \subseteq X$$
and $B_\delta(x)$ is convex.

**Definition 16** We say $f$ has a saddle at $x$ if $Df(x) = 0$ but $f$ has neither a local maximum nor a local minimum at $x$.

**Corollary 17** Suppose $X \subseteq \mathbb{R}^n$ is open and $x \in X$. If $f : X \rightarrow \mathbb{R}$ is $C^2$, then there is an orthonormal basis $\{v_1, \ldots, v_n\}$ and corresponding eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ of $D^2f(x)$ such that

$$f(x + h) = f(x + \gamma_1 v_1 + \cdots + \gamma_n v_n) = f(x) + \sum_{i=1}^n (Df(x)v_i) \gamma_i + \frac{1}{2} \sum_{i=1}^n \lambda_i \gamma_i^2 + o\left(|\gamma|^2\right)$$

where $\gamma_i = h \cdot v_i$.

1. If $f \in C^3$, we may strengthen $o\left(|\gamma|^2\right)$ to $O\left(|\gamma|^3\right)$.
2. If $f$ has a local maximum or local minimum at $x$, then

$$Df(x) = 0$$

3. If $Df(x) = 0$, then

$$\lambda_1, \ldots, \lambda_n > 0 \Rightarrow f \text{ has a local minimum at } x$$
$$\lambda_1, \ldots, \lambda_n < 0 \Rightarrow f \text{ has a local maximum at } x$$
$$\lambda_i < 0 \text{ for some } i, \lambda_j > 0 \text{ for some } j \Rightarrow f \text{ has a saddle at } x$$
$$\lambda_1, \ldots, \lambda_n \geq 0, \lambda_i > 0 \text{ for some } i \Rightarrow f \text{ has a local minimum or a saddle at } x$$
$$\lambda_1, \ldots, \lambda_n \leq 0, \lambda_i < 0 \text{ for some } i \Rightarrow f \text{ has a local maximum or a saddle at } x$$
$$\lambda_1 = \cdots = \lambda_n = 0 \text{ gives no information.}$$
**Proof:** (Sketch) From our study of quadratic forms, we know the behavior of the quadratic terms is determined by the signs of the eigenvalues. If \( \lambda_i = 0 \) for some \( i \), then we know that the quadratic form arising from the second partial derivatives is identically zero in the direction \( v_i \), and the higher derivatives will determine the behavior of the function \( f \) in the direction \( v_i \). For example, if \( f(x) = x^3 \), then \( f'(0) = 0, f''(0) = 0 \), but we know that \( f \) has a saddle at \( x = 0 \); however, if \( f(x) = x^4 \), then again \( f'(0) = 0 \) and \( f''(0) = 0 \) but \( f \) has a local (and global) minimum at \( x = 0 \).
Figure 1: The Mean Value Theorem.