Economics 204 Summer/Fall 2016
Lecture 14+ – Friday August 12, 2016

Differential Equations

Existence and Uniqueness of Solutions

Definition 1 A *differential equation* is an equation of the form

$$y'(t) = F(y(t), t)$$

where $F : U \to \mathbf{R}^n$ and U is an open subset of $\mathbf{R}^n \times \mathbf{R}$.

An *initial value problem* is a differential equation combined with an initial condition

$$y(t_0) = y_0$$

with $(y_0, t_0) \in U$.

A *solution* of the initial value problem is a differentiable function $y : (a, b) \to \mathbf{R}^n$ such that $t_0 \in (a, b)$, $y(t_0) = y_0$ and, for all $t \in (a, b)$, $\frac{dy}{dt} = F(y(t), t)$.

The *general solution* of the differential equation is the family of all solutions for all initial values $(y_0, t_0) \in U$.

Theorem 2 *Consider the initial value problem*

$$y'(t) = F(y(t), t),\ y(t_0) = y_0 \tag{1}$$

Let U be an open set in $\mathbf{R}^n \times \mathbf{R}$ containing (y_0, t_0).

1. *Suppose $F : U \to \mathbf{R}^n$ is continuous. Then the initial value problem has a solution.*
2. *If, in addition, F is Lipschitz in y on U, i.e. there is a constant K such that for all $(y, t), (\hat{y}, t) \in U$,*
$$|F(y, t) - F(\hat{y}, t)| \leq K|y - \hat{y}|$$
then there is an interval (a, b) containing t_0 such that the solution is unique on (a, b).

Proof: We consider only the case in which F is Lipschitz.

Since U is open, we may choose $r > 0$ such that

$$R = \{(y, t) : |y - y_0| \leq r, |t - t_0| \leq r\} \subseteq U$$

Since F is continuous, we may find $M \in \mathbf{R}$ such that $|F(y, t)| \leq M$ for all $(y, t) \in R$.

Given the Lipschitz condition, we may assume that

$$|F(y,t) - F(\hat{y},t)| \le K|y - \hat{y}| \text{ for all } (y,t), (\hat{y},t) \in R$$

Let

$$\delta = \min\left\{\frac{1}{2K}, \frac{r}{M}\right\}$$

We claim the initial value problem has a unique solution on $(t_0 - \delta, t_0 + \delta)$.

Let C be the space of continuous functions from $[t_0 - \delta, t_0 + \delta]$ to $\mathbf{R}^n$, endowed with the sup norm

$$\|f\|_\infty = \sup\{|f(t)| : t \in [t_0 - \delta, t_0 + \delta]\}$$

Let

$$S = \{z \in C : (z(s), s) \in R \text{ for all } s \in [t_0 - \delta, t_0 + \delta]\}$$

S is a closed subset of the complete metric space C, so S is a complete metric space.

Consider the function $I : S \to C$ defined by

$$I(z)(t) = y_0 + \int_{t_0}^{t} F(z(s), s)\, ds$$

$I(z)$ is defined and continuous because F is bounded and continuous on R. Observe that if $(z(s), s) \in R$ for all $s \in [t_0 - \delta, t_0 + \delta]$, then

$$\begin{aligned} |I(z)(t) - y_0| &= \left|\int_{t_0}^{t} F(z(s), s)\, ds\right| \\ &\le |t - t_0| \max\{|F(y,s)| : (y,s) \in R\} \\ &\le \delta M \\ &\le r \end{aligned}$$

so $(I(z)(t), t) \in R$ for all $t \in [t_0 - \delta, t_0 + \delta]$. Thus, $I : S \to S$.

Given two functions $x, z \in S$ and $t \in [t_0 - \delta, t_0 + \delta]$,

$$\begin{aligned} |I(z)(t) - I(x)(t)| &= \left|y_0 + \int_{t_0}^{t} F(z(s), s)\, ds - y_0 - \int_{t_0}^{t} F(x(s), s)\, ds\right| \\ &= \left|\int_{t_0}^{t} \left(F(z(s), s) - F(x(s), s)\right)\, ds\right| \\ &\le |t - t_0| \sup\{|F(z(s), s) - F(x(s), s)| : s \in [t_0 - \delta, t_0 + \delta]\} \\ &\le \delta K \sup\{|z(s) - x(s)| : s \in [t_0 - \delta, t_0 + \delta]\} \\ &\le \frac{1}{2}\|z - x\|_\infty \end{aligned}$$

Therefore, $\|I(z) - I(x)\|_\infty \le \frac{1}{2}\|z - x\|_\infty$, so I is a contraction. Since S is a complete metric space, I has a unique fixed point $y \in S$. Therefore, for all $t \in [t_0 - \delta, t_0 + \delta]$, we have

$$y(t) = y_0 + \int_{t_0}^{t} F(y(s), s)\, ds$$

F is continuous, so the Fundamental Theorem of Calculus implies that

$$y'(t) = F(y(t), t)$$

for all $t \in (t_0 - \delta, t_0 + \delta)$. Since we also have

$$y(t_0) = y_0 + \int_{t_0}^{t_0} F(y(s), s)\, ds = y_0$$

y (restricted to $(t_0 - \delta, t_0 + \delta)$) is a solution of the initial value problem (1).

On the other hand, suppose that $\hat{y}$ is any solution of the initial value problem (1) on $(t_0 - \delta, t_0 + \delta)$. It is easy to check that $(\hat{y}(s), s) \in R$ for all $s \in (t_0 - \delta, t_0 + \delta)$, so we have $|F(\hat{y}(s), s)| \leq M$; this implies that $\hat{y}$ has a extension to a continuous function (still denoted $\hat{y}$) in S. Since $\hat{y}$ is a solution of the initial value problem, the Fundamental Theorem of Calculus implies that $I(\hat{y}) = \hat{y}$. Since y is the unique fixed point of I, $\hat{y} = y$.■

Example: Consider the initial value problem

$$y'(t) = 1 + y^2(t), \ \ y(0) = 0$$

Here, we have $F(y, t) = 1 + y^2$ which is Lipschitz in y over $U = V \times \mathbf{R}$, provided that V is bounded, but not over all of $\mathbf{R} \times \mathbf{R}$. The theorem tells us that the initial value problem has a unique solution over some interval of times (a, b), with $0 \in (a, b)$.

We claim the unique solution is $y(t) = \tan t$. To see this, note that

$$\begin{aligned}
y'(t) &= \frac{d}{dt} \tan t \\
&= \frac{d}{dt} \frac{\sin t}{\cos t} \\
&= \frac{\cos t \cos t - \sin t(-\sin t)}{\cos^2 t} \\
&= \frac{\cos^2 t + \sin^2 t}{\cos^2 t} \\
&= 1 + \frac{\sin^2 t}{\cos^2 t} \\
&= 1 + \tan^2 t \\
&= 1 + (y(t))^2 \\
y(0) &= \tan 0 \\
&= 0
\end{aligned}$$

Notice that $y(t)$ is defined for $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, but

$$\lim_{t \to -\frac{\pi}{2}^+} y(t) = -\infty \text{ and } \lim_{t \to \frac{\pi}{2}^-} y(t) = \infty$$

Thus, the solution of the initial value problem cannot be extended beyond the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, because the solution "blows up" at $-\pi/2$ and $\pi/2$.

Example Consider the initial value problem

$$y'(t) = 2\sqrt{|y|},\ y(0) = 0 \tag{2}$$

The function $F(y,t) = 2\sqrt{|y|}$ is not locally Lipschitz in y at $y = 0$:

$$\begin{aligned} 2\sqrt{|y|} - 2\sqrt{0} &= 2\sqrt{|y|} \\ &= \frac{2}{\sqrt{|y|}}|y| \\ &= \frac{2}{\sqrt{|y|}}|y-0| \end{aligned}$$

which is not a bounded multiple of $|y - 0|$. Given any $\alpha \geq 0$, let

$$y_\alpha(t) = \begin{cases} 0 & \text{if } \ t \leq \alpha \\ (t-\alpha)^2 & \text{if } \ t \geq \alpha \end{cases}$$

We claim that y_α is a solution of the initial value problem (2) for every $\alpha \geq 0$. For $t < \alpha$, $y'_\alpha(t) = 0 = \sqrt{|0|} = 2\sqrt{|y_\alpha(t)|}$. For $t > \alpha$, $y'_\alpha(t) = 2(t-\alpha) = 2\sqrt{(t-\alpha)^2} = 2\sqrt{|y_\alpha(t)|}$. For $t = \alpha$,

$$\begin{aligned} \lim_{h\to 0^+} \frac{y_\alpha(\alpha+h) - y_\alpha(\alpha)}{h} &= \lim_{h\to 0^+} \frac{h^2}{h} = 0 \\ \lim_{h\to 0^-} \frac{y_\alpha(\alpha+h) - y_\alpha(\alpha)}{h} &= \lim_{h\to 0^-} \frac{0}{h} = 0 \end{aligned}$$

so $y'_\alpha(\alpha) = 0 = 2\sqrt{|y_\alpha(\alpha)|}$. Finally, $y_\alpha(0) = 0$, so y_α is a solution of the initial value problem (2), so we see the solution is decidedly not unique!

Remark: The initial value problem of Equation (1) has a solution defined on the interval

$$\left(\inf\left\{t : \forall s \in (t, t_0] \ \ (y(s), s) \in U\right\}, \sup\left\{t : \forall s \in [t_0, t) \ \ (y(s), s) \in U\right\}\right)$$

and it is unique on this interval provided that F is locally Lipschitz on U, i.e. for every $(y,t) \in U$, there is an open set V with $(y,t) \in V \subseteq U$ such that F is Lipschitz on V.

Autonomous Differential Equations

In many situations of interest, the function F in the differential equation does not depend on t.

Definition 3 An *autonomous differential equation* is a differential equation of the form

$$y'(t) = F(y(t))$$

where $F : \mathbf{R}^n \to \mathbf{R}^n$ depends on t only through the value of $y(t)$.

A *stationary point* of an autonomous differential equation is a point $y_s \in \mathbf{R}^n$ such that $F(y_s) = 0$.

We study the qualitative properties of autonomous differential equations by looking for stationary points. The constant function

$$y(t) = y_s$$

is a solution (and the unique solution when F is Lipschitz) of the initial value problem

$$y' = F(y),\ y(t_0) = y_s$$

If F is C^2, then Taylor's Theorem implies that near a stationary point y_s,

$$\begin{aligned} F(y_s + h) &= F(y_s) + DF(y_s)h + O\left(|h|^2\right) \\ &= DF(y_s)h + O\left(|h|^2\right) \end{aligned}$$

Thus, when we are sufficiently close to the stationary point, the solutions of the autonomous differential equation are closely approximated by the solutions of the *linear* differential equation

$$y' = (y - y_s)' = DF(y_s)(y - y_s)$$

Thus, we study solutions of linear differential equations, using linear algebra. For this we first recall the formulation of complex exponentials, which are central in the general solution of linear differential equations.

Complex Exponentials

Recall that the exponential function e^x (for $x \in \mathbf{R}$ or $x \in \mathbf{C}$) is given by the Taylor series

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

For $x, y \in \mathbf{C}$, we have

$$e^{x+y} = e^x e^y$$

If $x \in \mathbf{C}$, $x = a + ib$ for $a, b \in \mathbf{R}$, so

$$\begin{aligned} e^x &= e^{a+ib} \\ &= e^a e^{ib} \\ &= e^a \left(\sum_{k=0}^{\infty} \frac{(ib)^k}{k!} \right) \\ &= e^a \left(\sum_{k=0}^{\infty} \frac{(ib)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(ib)^{2k+1}}{(2k+1)!} \right) \\ &= e^a \left(\sum_{k=0}^{\infty} i^{2k} \frac{b^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} i^{2k} \frac{b^{2k+1}}{(2k+1)!} \right) \\ &= e^a \left(\sum_{k=0}^{\infty} (-1)^k \frac{b^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{b^{2k+1}}{(2k+1)!} \right) \\ &= e^a \left(\cos b + i \sin b \right) \end{aligned}$$

Now suppose that $t \in \mathbf{R}$, so

$$e^{tx} = e^{ta+itb} = e^{ta}(\cos tb + i \sin tb)$$

- If $a < 0$, then $e^{tx} \to 0$ as $t \to \infty$
- If $a > 0$, then $|e^{tx}| \to \infty$ as $t \to \infty$
- If $a = 0$, then $|e^{tx}| = 1$ for all $t \in \mathbf{R}$

Linear Differential Equations with Constant Coefficients

Let $M \in \mathbf{R}^{n \times n}$. The linear differential equation

$$y' = (y - y_s)' = M(y - y_s)$$

has a complete solution in closed form.

The matrix representation

$$M = DF(y_s)$$

need not be symmetric, hence may not be diagonalizable. If M *is* diagonalizable over $\mathbf{C}$, the complete solution takes the following simple form:

Theorem 4 *Consider the linear differential equation*

$$y' = (y - y_s)' = M(y - y_s)$$

where M is a real $n \times n$ matrix. Suppose that M can be diagonalized over the complex field $\mathbf{C}$. Let U be the standard basis of $\mathbf{R}^n$ and $V = \{v_1, \ldots, v_n\}$ be a basis of (complex) eigenvectors corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbf{C}$. Then the solution of the initial value problem is given by

$$y(t) = y_s + P^{-1} \begin{pmatrix} e^{\lambda_1(t-t_0)} & 0 & 0 & \cdots & 0 \\ 0 & e^{\lambda_2(t-t_0)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{\lambda_n(t-t_0)} \end{pmatrix} P(y(t_0) - y_s) \tag{3}$$

where $P = (Mtx)_{V,U}(id)$, and the general complex solution is obtained by allowing $y(t_0)$ to vary over $\mathbf{C}^n$; it has n complex degrees of freedom. The general real solution is obtained by allowing $y(t_0)$ to vary over $\mathbf{R}^n$; it has n real degrees of freedom. Every real solution is a linear combination of the real and imaginary parts of a complex solution. In particular,

1. *If the real part of each eigenvalue is less than zero, all solutions converge to y_s.*
2. *If the real part of each eigenvalue is greater than zero, all solutions diverge from y_s and tend to infinity.*

3. *If the real parts of some eigenvalues are less than zero and the real parts of other eigenvalues are greater than zero, solutions follow roughly hyperbolic paths.*

4. *If the real parts of all eigenvalues are zero, all solutions follow closed cycles around y_s.*

Remark: If one or more of the eigenvalues are complex, each of the three matrices in Equation (3) will contain complex entries, but the *product* of the three matrices is *real*. Thus, if the initial condition y_0 is real, Equation (3) gives us a real solution; indeed, it gives us the unique solution of the initial value problem.

Remark: Given a fixed time t_0, the general real solution is obtained by varying the initial values of $y(t_0)$ over $\mathbf{R}^n$, which provides n real degrees of freedom. You might think that varying t_0 provides one additional degree of freedom, but it doesn't. Given any solution satisfying the initial condition $y(t_0) = y_0$, the solution is defined on some interval $(t_0 - \delta, t_0 + \delta)$; given $t_1 \in (t_0 - \delta, t_0 + \delta)$, let $y_1 = y(t_1)$; then the solution with initial condition $y(t_1) = y_1$ is the same as the solution with initial condition $y(t_0) = y_0$. The same holds true for the general complex solution.

Proof: Let $P = (Mtx)_{V,U}(id)$. Rewrite the differential equation in terms of a new variable

$$z = Py$$

the representation of the solution with respect to the basis V of eigenvectors. Let $z_s = Py_s$. Then we have

$$\begin{aligned}
z - z_s &= P(y - y_s) \\
(z - z_s)' &= z' \\
&= Py' \\
&= PM(y - y_s) \\
&= PMP^{-1}(z - z_s) \\
&= B(z - z_s)
\end{aligned}$$

where

$$B = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Thus, the i^{th} component of $(z(t) - z_s)$ satisfies the differential equation

$$(z(t) - z_s)'_i = \lambda_i \left(z(t) - z_s\right)_i$$

so

$$(z(t) - z_s)_i = e^{\lambda_i (t - t_0)}(z(t_0) - z_s)_i$$

so

$$z(t) - z_s = \begin{pmatrix} e^{\lambda_1(t-t_0)} & 0 & 0 & \cdots & 0 \\ 0 & e^{\lambda_2(t-t_0)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{\lambda_n(t-t_0)} \end{pmatrix} (z(t_0) - z_s)$$

$$\begin{aligned} y(t) - y_s &= P^{-1}(z(t) - z_s) \\ &= P^{-1} \begin{pmatrix} e^{\lambda_1(t-t_0)} & 0 & 0 & \cdots & 0 \\ 0 & e^{\lambda_2(t-t_0)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{\lambda_n(t-t_0)} \end{pmatrix} (z(t_0) - z_s) \\ &= P^{-1} \begin{pmatrix} e^{\lambda_1(t-t_0)} & 0 & 0 & \cdots & 0 \\ 0 & e^{\lambda_2(t-t_0)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{\lambda_n(t-t_0)} \end{pmatrix} P(y(t_0) - y_s) \end{aligned}$$

■

The Form of Real Solutions

We can determine the form of the real solutions once we know the eigenvalues. In an important special case, we can solve for the solution of the initial value problem without calculating the diagonalization, as in Equation (3).

Theorem 5 *Consider the differential equation*

$$y' = (y - y_s)' = M(y - y_s)$$

Suppose that the matrix M can be diagonalized over $\mathbf{C}$. *Let the eigenvalues of M with the correct multiplicity be*

$$a_1 + ib_1, a_1 - ib_1, \ldots, a_m + ib_m, a_m - ib_m, a_{m+1}, \ldots, a_{n-m}$$

Then for each fixed $i = 1, \ldots, n$, every real solution is of the form

$$(y(t) - y_s)_i = \sum_{j=1}^{m} e^{a_j(t-t_0)} \left(C_{ij} \cos b_j(t - t_0) + D_{ij} \sin b_j(t - t_0)\right) + \sum_{j=m+1}^{n-m} C_{ij} e^{a_j(t-t_0)}$$

The n^2 parameters

$$\{C_{ij} : i = 1, \ldots, n; j = 1, \ldots, n - m\} \cup \{D_{ij} : i = 1, \ldots, n; j = 1, \ldots, m\}$$

have n real degrees of freedom. The parameters are uniquely determined from the n real initial conditions of an initial value problem.

Proof: Rewrite the expression for the solution y as

$$(y(t) - y_s)_i = \sum_{j=1}^{n} \gamma_{ij} e^{\lambda_j (t-t_0)}$$

Recall that the non-real eigenvalues occur in conjugate pairs, so suppose that

$$\lambda_j = a + ib, \ \lambda_k = a - ib$$

so the expression for $(y(t) - y_s)_i$ contains the pair of terms

$$\begin{aligned}
\gamma_{ij} e^{\lambda_j (t-t_0)} + \gamma_{ik} e^{\lambda_k (t-t_0)} &= \gamma_{ij} e^{a(t-t_0)} \left(\cos b(t-t_0) + i \sin b(t-t_0) \right) \\
&\quad + \gamma_{ik} e^{a(t-t_0)} \left(\cos b(t-t_0) - i \sin b(t-t_0) \right) \\
&= e^{a(t-t_0)} \left((\gamma_{ij} + \gamma_{ik}) \cos b(t-t_0) + i \left(\gamma_{ij} - \gamma_{ik} \right) \sin b(t-t_0) \right) \\
&= e^{a(t-t_0)} \left(C_{ij} \cos b(t-t_0) + D_{ij} \sin b(t-t_0) \right)
\end{aligned}$$

Since this must be real for all t, we must have

$$C_{ij} = \gamma_{ij} + \gamma_{ik} \in \mathbf{R} \text{ and } D_{ij} = i \left(\gamma_{ij} - \gamma_{ik} \right) \in \mathbf{R}$$

so γ_{ij} and γ_{ik} are complex conjugates; this can also be shown directly from the matrix formula for y in terms of z.

Thus, if the eigenvalues $\lambda_1, \ldots, \lambda_n$ are

$$a_1 + ib_1, a_1 - ib_1, a_2 + ib_2, a_2 - ib_2, \ldots, a_m + ib_m, a_m - ib_m, a_{m+1}, \ldots, a_{n-m}$$

every real solution will be of the form

$$(y(t) - y_s)_i = \sum_{j=1}^{m} e^{a_j (t-t_0)} \left(C_{ij} \cos b_j (t-t_0) + D_{ij} \sin b_j (t-t_0) \right) + \sum_{j=m+1}^{n-m} C_{ij} e^{a_j (t-t_0)}$$

Since the differential equation satisfies a Lipschitz condition, the initial value problem has a unique solution determined by the n real initial conditions. Thus, the general solution has exactly n real degrees of freedom in the n^2 coefficients. ∎

Remark: The constraints among the coefficients C_{ij}, D_{ij} can be complicated. One cannot just solve for the coefficients of y_1 from the initial conditions, then derive the coefficients for $y_2, \ldots, y_n$. For example, consider the differential equation

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

The eigenvalues are 2 and 1. If we set

$$\begin{aligned}
y_1(t) &= C_{11} e^{2(t-t_0)} + C_{12} e^{t-t_0} \\
y_2(t) &= C_{21} e^{2(t-t_0)} + C_{22} e^{t-t_0}
\end{aligned}$$

we get

$$\begin{aligned} y_1(t_0) &= C_{11} + C_{12} \\ y_2(t_0) &= C_{21} + C_{22} \end{aligned}$$

which doesn't have a unique solution. However, from the original differential equation, we have

$$y_1(t) = y_1(t_0)e^{2(t-t_0)},\ y_2(t) = y_2(t_0)e^{t-t_0}$$

so

$$\begin{array}{cc} C_{11} = y_1(t_0) & C_{12} = 0 \\ C_{21} = 0 & C_{22} = y_2(t_0) \end{array}$$

One *can* find the solution to the initial value problem by plugging the n real initial conditions into Equation (3) in Theorem 4 above, and the general solution by varying the initial conditions.

However, in the special case

$$\bar{y} = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}$$

the coefficients

$$C_{11}, \ldots, C_{1n-m}, D_{11}, \ldots, D_{1m}$$

in the general solution are arbitrary real numbers; once they are set, the other coefficients are determined. Write

$$y(t) - y_s = \sum_{j=1}^{m} e^{a_j(t-t_0)} \left(C_j \cos b_j(t-t_0) + D_j \sin b_j(t-t_0)\right) + \sum_{j=m+1}^{n-m} C_j e^{a_j(t-t_0)}$$

For the initial value problem, compute the first $n-1$ derivatives of y at t_0 and set them equal to the initial conditions. This yields n linear equations in the n coefficients, which have a unique solution. See the next section for an example of this.

Note also that

$$\begin{aligned} C_j e^{a_j(t-t_0)} &= \left(C_j e^{-a_j t_0}\right) e^{a_j t} \\ \cos b_j(t-t_0) &= \cos\left(b_j t - b_j t_0\right) \\ &= \cos b_j t \cos b_j t_0 + \sin b_j t \sin b_j t_0 \\ \sin b_j(t-t_0) &= \sin\left(b_j t - b_j t_0\right) \\ &= -\cos b_j t \sin b_j t_0 + \sin b_j t \cos b_j t_0 \end{aligned}$$

so we can also write

$$y(t) - y_s = \sum_{j=1}^{m} e^{a_j t} \left(C_j \cos b_j t + D_j \sin b_j t\right) + \sum_{j=m+1}^{n-m} C_j e^{a_j t}$$

Second Order Linear Differential Equations

Consider the second order differential equation $y'' = cy + by'$ with $b, c \in \mathbf{R}$.

Rewrite this as a *first order* linear differential equation in two variables:

$$\begin{aligned}
\bar{y}(t) &= \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} \\
\bar{y}'(t) &= \begin{pmatrix} y'(t) \\ y''(t) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ c & b \end{pmatrix} \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ c & b \end{pmatrix} \bar{y}
\end{aligned}$$

The eigenvalues are $\frac{b \pm \sqrt{b^2+4c}}{2}$, the roots of the equation $\lambda^2 - b\lambda - c = 0$. The qualitative behavior of the solutions can be explicitly described from the coefficients b and c, by determining whether the eigenvalues are real or complex, and whether the real parts are negative, zero, or positive. See Section 6 of the Differential Equations Handout.

Example Consider the second order linear differential equation

$$y'' = 2y + y'$$

As above, let

$$\bar{y} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

so the equation becomes

$$\bar{y}' = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \bar{y}$$

The eigenvalues are the roots of the characteristic polynomial

$$\lambda^2 - \lambda - 2 = 0$$

Eigenvalues and corresponding eigenvectors are given by

$$\begin{aligned}
\lambda_1 = 2 &\quad v_1 = (1, 2) \\
\lambda_2 = -1 &\quad v_2 = (1, -1)
\end{aligned}$$

Note that because the matrix

$$M = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$$

has two distinct eigenvalues (alternatively, because the eigenvectors $\{v_1, v_2\}$ form a basis), M is diagonalizable. From this information alone, we know the qualitative properties of the solutions are as given in the phase plane diagram (see Figure 1):

- Solutions are roughly hyperbolic in shape with asymptotes along the eigenvectors. Along the eigenvector v_1, the solutions flow off to infinity; along the eigenvector v_2, the solutions converge to zero.

- Solutions flow in directions consistent with flows along asymptotes

- On the y-axis, we have $y' = 0$, which means that everywhere on the y-axis (except at the stationary point 0), the solution must have a vertical tangent.

- On the y'-axis, we have $y = 0$, so we have

$$y'' = 2y + y' = y'$$

  Thus, above the y-axis, $y'' = y' > 0$, so y' is increasing along the direction of the solution; below the y-axis, $y'' = y' < 0$, so y' is decreasing along the direction of the solution.

- Along the line $y' = -2y$, $y'' = 2y - 2y = 0$, so y' achieves a minimum or maximum where it crosses that line.

The general solution is given by

$$\begin{aligned}
\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} &= Mtx_{U,V}(id) \begin{pmatrix} e^{2(t-t_0)} & 0 \\ 0 & e^{-(t-t_0)} \end{pmatrix} Mtx_{V,U}(id) \begin{pmatrix} y(t_0) \\ y'(t_0) \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} e^{2(t-t_0)} & 0 \\ 0 & e^{-(t-t_0)} \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{pmatrix} \begin{pmatrix} y(t_0) \\ y'(t_0) \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \frac{e^{2(t-t_0)}}{3} & \frac{e^{2(t-t_0)}}{3} \\ \frac{2e^{-(t-t_0)}}{3} & -\frac{e^{-(t-t_0)}}{3} \end{pmatrix} \begin{pmatrix} y(t_0) \\ y'(t_0) \end{pmatrix} \\
&= \begin{pmatrix} \frac{e^{2(t-t_0)}+2e^{-(t-t_0)}}{3} & \frac{e^{2(t-t_0)}-e^{-(t-t_0)}}{3} \\ \frac{2e^{2(t-t_0)}-2e^{-(t-t_0)}}{3} & \frac{2e^{2(t-t_0)}+e^{-(t-t_0)}}{3} \end{pmatrix} \begin{pmatrix} y(t_0) \\ y'(t_0) \end{pmatrix} \\
&= \begin{pmatrix} \frac{y(t_0)+y'(t_0)}{3} e^{2(t-t_0)} + \frac{2y(t_0)-y'(t_0)}{3} e^{-(t-t_0)} \\ \frac{2y(t_0)+2y'(t_0)}{3} e^{2(t-t_0)} + \frac{-2y(t_0)+y'(t_0)}{3} e^{-(t-t_0)} \end{pmatrix}
\end{aligned}$$

The general solution has two real degrees of freedom; a specific solution is determined by specifying initial conditions $y(t_0)$ and $y'(t_0)$.

Because

$$\bar{y} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

it is easier to find the general solution by first setting

$$y(t) = C_1 e^{2(t-t_0)} + C_2 e^{-(t-t_0)}$$

Then observe

$$y'(t) = 2C_1e^{2(t-t_0)} - C_2e^{-(t-t_0)}$$

Then evaluating $y(t)$ and $y'(t)$ at $t = t_0$ and using the initial conditions $y(t_0)$ and $y'(t_0)$ gives two equations involving C_1 and C_2, which can be solved for C_1 and C_2 to determine the general solution.

Thus

$$\begin{aligned}
y(t_0) &= C_1 + C_2 \\
y'(t_0) &= 2C_1 - C_2 \\
C_1 &= \frac{y(t_0) + y'(t_0)}{3} \\
C_2 &= \frac{2y(t_0) - y'(t_0)}{3}
\end{aligned}$$

Thus the general solution is

$$y(t) = \frac{y(t_0) + y'(t_0)}{3}e^{2(t-t_0)} + \frac{2y(t_0) - y'(t_0)}{3}e^{-(t-t_0)}$$

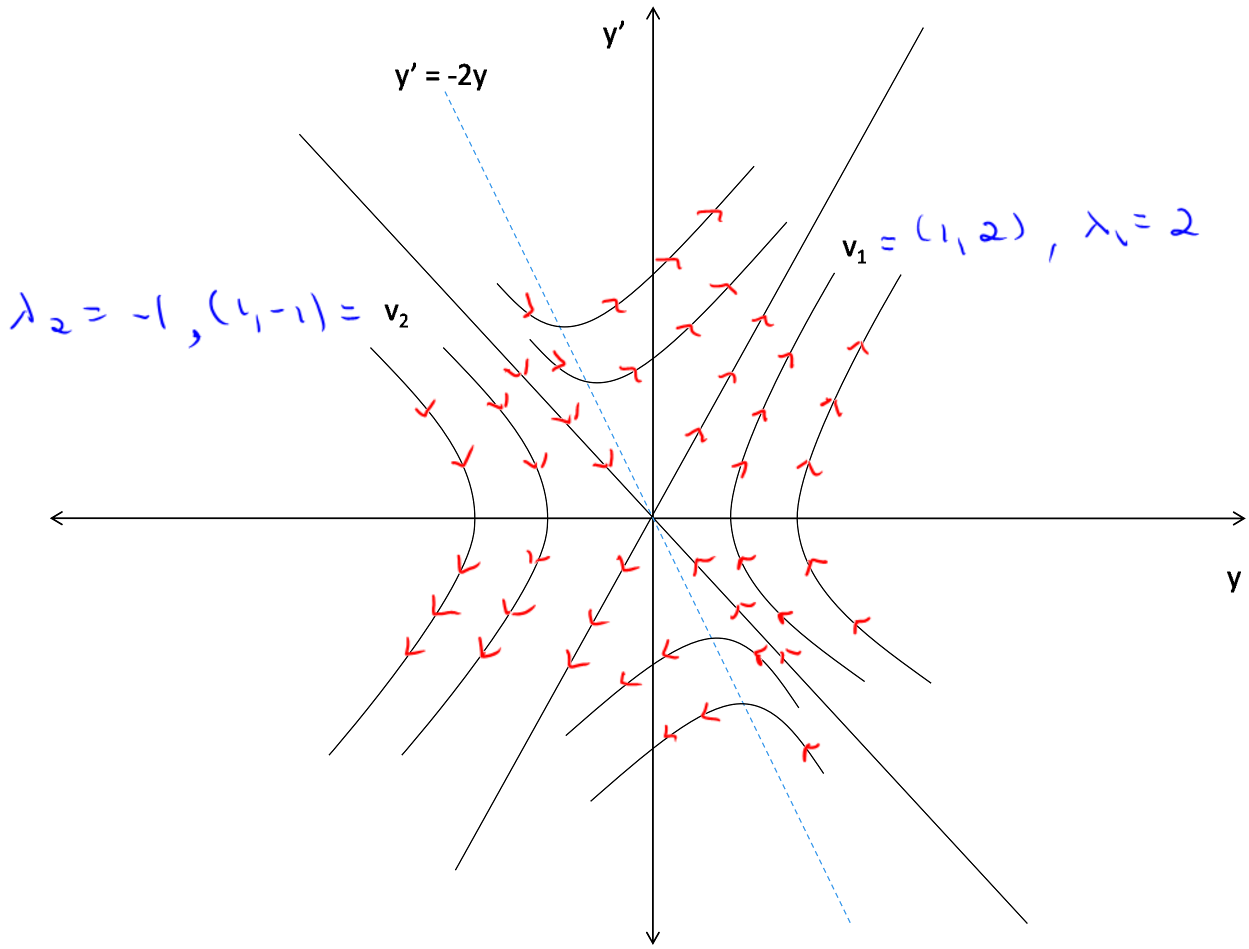

Figure 1: Phase plane diagram for $y'' = 2y + y'$.