Section 2.4. Open and Closed Sets

Definition 1 Let \((X, d)\) be a metric space. A set \(A \subseteq X\) is open if
\[
\forall x \in A \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq A
\]
A set \(C \subseteq X\) is closed if \(X \setminus C\) is open.

See Figure 1.

Example: \((a, b)\) is open in the metric space \(E^1\) (\(\mathbb{R}\) with the usual Euclidean metric). Given \(x \in (a, b), a < x < b\). Let\[
\varepsilon = \min\{x - a, b - x\} > 0
\]
Then
\[
y \in B_\varepsilon(x) \Rightarrow y \in (x - \varepsilon, x + \varepsilon) \\
\subseteq (x - (x - a), x + (b - x)) \\
= (a, b)
\]
so \(B_\varepsilon(x) \subseteq (a, b)\), so \((a, b)\) is open.

Notice that \(\varepsilon\) depends on \(x\); in particular, \(\varepsilon\) gets smaller as \(x\) nears the boundary of the set.

Example: In \(E^1\), \([a, b]\) is closed. \(\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)\) is a union of two open sets, which must be open.

Example: In the metric space \([0, 1]\), \([0, 1]\) is open. With \([0, 1]\) as the underlying metric space, \(B_\varepsilon(0) = \{x \in [0, 1] : |x - 0| < \varepsilon\} = [0, \varepsilon]\).

Thus, openness and closedness depend on the underlying metric space as well as on the set.

Example: Most sets are neither open nor closed. For example, in \(E^1\), \([0, 1] \cup (2, 3)\) is neither open nor closed.

Example: An open set may consist of a single point. For example, if \(X = \mathbb{N}\) and \(d(m, n) = |m - n|\), then \(B_{1/2}(1) = \{m \in \mathbb{N} : |m - 1| < 1/2\} = \{1\}\). Since 1 is the only element of the set \(\{1\}\) and \(B_{1/2}(1) = \{1\} \subseteq \{1\}\), the set \(\{1\}\) is open.

Example: In any metric space \((X, d)\) both \(\emptyset\) and \(X\) are open, and both \(\emptyset\) and \(X\) are closed. To see that \(\emptyset\) is open, note that the statement
\[
\forall x \in \emptyset \exists \varepsilon > 0 \ B_\varepsilon(x) \subseteq \emptyset
\]
is vacuously true since there aren’t any \( x \in \emptyset \). To see that \( X \) is open, note that since \( B_\varepsilon(x) \) is by definition \( \{ z \in X : d(z, x) < \varepsilon \} \), it is trivially contained in \( X \). Since \( \emptyset \) is open, \( X \) is closed; since \( X \) is open, \( \emptyset \) is closed.

**Example:** Open balls are open sets. Suppose \( y \in B_\varepsilon(x) \). Then \( d(x, y) < \varepsilon \). Let \( \delta = \varepsilon - d(x, y) > 0 \). If \( d(z, y) < \delta \), then

\[
\begin{align*}
d(z, x) & \leq d(z, y) + d(y, x) \\
& < \delta + d(x, y) \\
& = \varepsilon - d(x, y) + d(x, y) \\
& = \varepsilon
\end{align*}
\]

so \( B_\delta(y) \subseteq B_\varepsilon(x) \), so \( B_\varepsilon(x) \) is open.

**Theorem 2 (Thm. 4.2)** Let \((X, d)\) be a metric space. Then

1. \( \emptyset \) and \( X \) are both open, and both closed.
2. The union of an arbitrary (finite, countable, or uncountable) collection of open sets is open.
3. The intersection of a finite collection of open sets is open.

**Proof:**

1. We have already shown this.
2. Suppose \( \{A_\lambda\}_{\lambda \in \Lambda} \) is a collection of open sets.

\[
x \in \bigcup_{\lambda \in \Lambda} A_\lambda \implies \exists \lambda_0 \in \Lambda \text{ s.t. } x \in A_{\lambda_0} \implies \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq A_{\lambda_0} \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda
\]

so \( \bigcup_{\lambda \in \Lambda} A_\lambda \) is open.
3. Suppose \( A_1, \ldots, A_n \subseteq X \) are open sets. If \( x \in \bigcap_{i=1}^n A_i \), then

\[
x \in A_1, x \in A_2, \ldots, x \in A_n
\]

so

\[
\exists \varepsilon_1, \ldots, \varepsilon_n > 0 \text{ s.t. } B_{\varepsilon_1}(x) \subseteq A_1, \ldots, B_{\varepsilon_n}(x) \subseteq A_n
\]

Let

\[
\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\} > 0
\]
(Note this is where we need the fact that we are taking a finite intersection. The infimum of an infinite set of positive numbers could be zero. And the intersection of an infinite collection of open sets need not be open.)

Then
\[ B_\varepsilon(x) \subseteq B_{\varepsilon_1}(x) \subseteq A_1, \ldots, B_\varepsilon(x) \subseteq B_{\varepsilon_n}(x) \subseteq A_n \]

so
\[ B_\varepsilon(x) \subseteq \bigcap_{i=1}^{n} A_i \]

which proves that \( \cap_{i=1}^{n} A_i \) is open.

\[ \square \]

**Definition 3**

- The **interior** of \( A \), denoted \( \text{int} A \), is the largest open set contained in \( A \) (the union of all open sets contained in \( A \)).

- The **closure** of \( A \), denoted \( \overline{A} \), is the smallest closed set containing \( A \) (the intersection of all closed sets containing \( A \)).

- The **exterior** of \( A \), denoted \( \text{ext} A \), is the largest open set contained in \( X \setminus A \).

- The **boundary** of \( A \), denoted \( \partial A = (X \setminus A) \cap \overline{A} \)

**Example:** Let \( A = [0, 1] \cup (2, 3) \). Then

\[
\begin{align*}
\text{int} A &= (0, 1) \cup (2, 3) \\
\overline{A} &= [0, 1] \cup [2, 3] \\
\text{ext} A &= \text{int} (X \setminus A) \\
&= (-\infty, 0) \cup (1, 2) \cup (3, +\infty) \\
\partial A &= (X \setminus A) \cap \overline{A} \\
&= ((-\infty, 0] \cup [1, 2] \cup [3, +\infty)) \cap ([0, 1] \cup [2, 3]) \\
&= \{0, 1, 2, 3\}
\end{align*}
\]

**Theorem 4 (Thm. 4.13)** A set \( A \) in a metric space \((X, d)\) is closed if and only if

\[ \{x_n\} \subseteq A, x_n \to x \in X \Rightarrow x \in A \]

**Proof:** Suppose \( A \) is closed. Then \( X \setminus A \) is open. Consider a convergent sequence \( x_n \to x \in X \), with \( x_n \in A \) for all \( n \). If \( x \not\in A \), \( x \in X \setminus A \), so there is some \( \varepsilon > 0 \) such that \( B_\varepsilon(x) \subseteq X \setminus A \). (See Figure 2.) Since \( x_n \to x \), there exists \( N(\varepsilon) \) such that

\[ n > N(\varepsilon) \Rightarrow x_n \in B_\varepsilon(x) \]

\[ \Rightarrow x_n \in X \setminus A \]

\[ \Rightarrow x_n \not\in A \]

\[ ^1\text{This is different from the proof in de la Fuente: he puts the meat of the proof into Theorem 4.12} \]
contradiction. Therefore,
\[ x_n \subset A, x_n \to x \in X \Rightarrow x \in A \]

Conversely, suppose
\[ \{x_n\} \subset A, x_n \to x \in X \Rightarrow x \in A \]

We need to show that \( A \) is closed, i.e. \( X \setminus A \) is open. Suppose not, so \( X \setminus A \) is not open. Then there exists \( x \in X \setminus A \) such that for every \( \varepsilon > 0 \),
\[ B_{\varepsilon}(x) \not\subseteq X \setminus A \]
so there exists \( y \in B_{\varepsilon}(x) \) such that \( y \notin X \setminus A \). Then \( y \in A \), hence
\[ B_{\varepsilon}(x) \cap A \neq \emptyset \]

See Figure 3. Construct a sequence \( \{x_n\} \) as follows: for each \( n \), choose \( x_n \in B_{\frac{1}{n}}(x) \cap A \). Given \( \varepsilon > 0 \), we can find \( N(\varepsilon) \) such that \( N(\varepsilon) > \frac{1}{\varepsilon} \) by the Archimedean Property, so \( n > N(\varepsilon) \Rightarrow \frac{1}{n} < \frac{1}{N(\varepsilon)} < \varepsilon \), so \( x_n \to x \). Then \( \{x_n\} \subset A, x_n \to x \), so \( x \in A \), contradiction. Therefore, \( X \setminus A \) is open, so \( A \) is closed. ■

Section 2.5. Limits of Functions

Note: Read this section of de la Fuente on your own.

Note that we may have \( \lim_{x \to a} f(x) = y \) even though

- \( f \) is not defined at \( a \); or
- \( f \) is defined at \( a \) but \( f(a) \neq y \).

The existence and value of the limit depends on values of \( f \) near \( a \) but not at \( a \).

Section 2.6. Continuity in Metric Spaces

**Definition 5** Let \( (X, d) \) and \( (Y, \rho) \) be metric spaces. A function \( f : X \to Y \) is **continuous at a point** \( x_0 \in X \) if \( \forall \varepsilon > 0 \ \exists \delta(x_0, \varepsilon) > 0 \) s.t. \( d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon \).

\( f \) is **continuous** if it is continuous at every element of its domain.

Note that \( \delta \) depends on \( x_0 \) and \( \varepsilon \).

This is a straightforward generalization of the definition of continuity in \( \mathbb{R} \). Continuity at \( x_0 \) requires:

- \( f(x_0) \) is defined; and
• either
  
  - \( x_0 \) is an isolated point of \( X \), i.e. \( \exists \varepsilon > 0 \) s.t. \( B_\varepsilon(x) = \{x\} \); or
  - \( \lim_{x \to x_0} f(x) \) exists and equals \( f(x_0) \)

Suppose \( f : X \to Y \) and \( A \subseteq Y \). Define \( f^{-1}(A) = \{x \in X : f(x) \in A\} \).

**Theorem 6 (Thm. 6.14)** Let \((X, d)\) and \((Y, \rho)\) be metric spaces, and \( f : X \to Y \). Then \( f \) is continuous if and only if

\[
f^{-1}(A) \text{ is open in } X \quad \forall A \subseteq Y \text{ s.t. } A \text{ is open in } Y
\]

**Proof:** Suppose \( f \) is continuous. Given \( A \subseteq Y \), \( A \) open, we must show that \( f^{-1}(A) \) is open in \( X \). Suppose \( x_0 \in f^{-1}(A) \). Let \( y_0 = f(x_0) \in A \). Since \( A \) is open, we can find \( \varepsilon > 0 \) such that \( B_\varepsilon(y_0) \subseteq A \). Since \( f \) is continuous, there exists \( \delta > 0 \) such that

\[
d(x, x_0) < \delta \quad \Rightarrow \quad \rho(f(x), f(x_0)) < \varepsilon
\]

\[
\Rightarrow \quad f(x) \in B_\varepsilon(y_0)
\]

\[
\Rightarrow \quad f(x) \in A
\]

\[
\Rightarrow \quad x \in f^{-1}(A)
\]

so \( B_\delta(x_0) \subseteq f^{-1}(A) \), so \( f^{-1}(A) \) is open. (See Figure 4.)

Conversely, suppose \( f^{-1}(A) \) is open in \( X \) \( \forall A \subseteq Y \) s.t. \( A \) is open in \( Y \)

We need to show that \( f \) is continuous. Let \( x_0 \in X, \varepsilon > 0 \). Let \( A = B_\varepsilon(f(x_0)) \). \( A \) is an open ball, hence an open set, so \( f^{-1}(A) \) is open in \( X \). \( x_0 \in f^{-1}(A) \), so there exists \( \delta > 0 \) such that \( B_\delta(x_0) \subseteq f^{-1}(A) \). (See Figure 5.)

\[
d(x, x_0) < \delta \quad \Rightarrow \quad x \in B_\delta(x_0)
\]

\[
\Rightarrow \quad x \in f^{-1}(A)
\]

\[
\Rightarrow \quad f(x) \in A
\]

\[
\Rightarrow \quad \rho(f(x), f(x_0)) < \varepsilon
\]

Thus, we have shown that \( f \) is continuous at \( x_0 \); since \( x_0 \) is an arbitrary point in \( X \), \( f \) is continuous.  

**Theorem 7 (Slightly weaker version of Thm. 6.10)** Let \((X, d_X)\), \((Y, d_Y)\) and \((Z, d_Z)\) be metric spaces. If \( f : X \to Y \) and \( g : Y \to Z \) are continuous, then \( g \circ f : X \to Z \) is continuous.

\[\text{2We give a direct proof; de la Fuente works via closed sets.}\]
**Proof:** Suppose \( A \subseteq Z \) is open. Since \( g \) is continuous, \( g^{-1}(A) \) is open in \( Y \); since \( f \) is continuous, \( f^{-1}(g^{-1}(A)) \) is open in \( X \).

We claim that

\[
f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)
\]

Observe

\[
x \in f^{-1}(g^{-1}(A)) \iff f(x) \in g^{-1}(A) \\
\iff g(f(x)) \in A \\
\iff (g \circ f)(x) \in A \\
\iff x \in (g \circ f)^{-1}(A)
\]

which establishes the claim. This shows that \( (g \circ f)^{-1}(A) \) is open in \( X \), so \( g \circ f \) is continuous.

\[\blacksquare\]

**Definition 8** [Uniform Continuity] Suppose \( f : (X, d) \rightarrow (Y, \rho) \). \( f \) is **uniformly continuous** if

\[
\forall \varepsilon > 0 \ \exists \delta(\varepsilon) > 0 \ \text{s.t.} \ \forall x_0 \in X, \ d(x, x_0) < \delta(\varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon
\]

Notice the important contrast with continuity: \( f \) is continuous means

\[
\forall x_0 \in X, \varepsilon > 0 \ \exists \delta(x_0, \varepsilon) > 0 \ \text{s.t.} \ d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon
\]

**Example:** Consider

\[
f(x) = \frac{1}{x}, \ x \in (0, 1]\]

\( f \) is continuous (why?). We will show that \( f \) is **not** uniformly continuous. Fix \( \varepsilon > 0 \) and \( x_0 \in (0, 1] \). If \( x = \frac{x_0}{1+\varepsilon x_0} \), then

\[
1 + \varepsilon x_0 > 1
\]

\[
x = \frac{x_0}{1 + \varepsilon x_0} < x_0
\]

\[
\frac{1}{x} - \frac{1}{x_0} > 0
\]

\[
|f(x) - f(x_0)| = \left| \frac{1}{x} - \frac{1}{x_0} \right|
\]

\[
= \frac{1}{x} - \frac{1}{x_0}
\]

\[
= \frac{1 + \varepsilon x_0}{x_0} - \frac{1}{x_0}
\]

\[
= \frac{\varepsilon x_0}{x_0}
\]

\[
= \varepsilon
\]
Thus, $\delta(x_0, \varepsilon)$ must be chosen small enough so that

$$\left| \frac{x_0}{1 + \varepsilon x_0} - x_0 \right| \geq \delta(x_0, \varepsilon)$$

$$\delta(x_0, \varepsilon) \leq x_0 - \frac{x_0}{1 + \varepsilon x_0}$$

$$= \frac{\varepsilon(x_0)^2}{1 + \varepsilon x_0}$$

$$< \varepsilon(x_0)^2$$

which converges to zero as $x_0 \to 0$. (See Figure 6.) So there is no $\delta(\varepsilon)$ that will work for all $x_0 \in (0, 1]$.

**Example:** If $f : \mathbb{R} \to \mathbb{R}$ and $f'(x)$ is defined and uniformly bounded on an interval $[a, b]$, then $f(x)$ is uniformly continuous on $[a, b]$. However, even a function with an unbounded derivative may be uniformly continuous. Consider

$$f(x) = \sqrt{x}, \ x \in [0, 1]$$

$f$ is continuous (why?). We will show that $f$ is uniformly continuous. Given $\varepsilon > 0$, let $\delta = \varepsilon^2$. Then given any $x_0 \in [0, 1]$, $|x - x_0| < \delta$ implies by the Fundamental Theorem of Calculus

$$|f(x) - f(x_0)| = \left| \int_{x_0}^{x} \frac{1}{2\sqrt{t}} \, dt \right|$$

$$\leq \int_{0}^{|x-x_0|} \frac{1}{2\sqrt{t}} \, dt$$

$$= \sqrt{|x-x_0|}$$

$$< \sqrt{\delta}$$

$$= \sqrt{\varepsilon^2}$$

$$= \varepsilon$$

Thus, $f$ is uniformly continuous on $[0, 1]$, even though $f'(x) \to \infty$ as $x \to 0$.

**Definition 9** Let $X, Y$ be normed vector spaces, $E \subseteq X$. $f : X \to Y$ is **Lipschitz on $E$** if

$$\exists K > 0 \text{ s.t. } \|f(x) - f(z)\|_Y \leq K\|x - z\|_X \ \forall x, z \in E$$

$f$ is **locally Lipschitz on $E$** if

$$\forall x_0 \in E \ \exists \varepsilon > 0 \text{ s.t. } f \text{ is Lipschitz on } B_{\varepsilon}(x_0) \cap E$$

7
Remark: de la Fuente only defines Lipschitz and locally Lipschitz in the context of normed vector spaces. The notions can also be defined analogously in metric spaces as follows: Let $(X, d)$ and $(Y, ρ)$ be metric spaces, $E ⊆ X$. $f : X → Y$ is Lipschitz on $E$ if

$$\exists K > 0 \text{ s.t. } ρ(f(x), f(z)) \leq Kd(x, z) \ \forall x, z ∈ E$$

Similarly, $f$ is locally Lipschitz on $E$ if

$$∀x_0 ∈ E \exists ε > 0 \text{ s.t. } f \text{ is Lipschitz on } B_ε(x_0) \cap E$$

Lipschitz continuity is stronger than either continuity or uniform continuity:

- locally Lipschitz $⇒$ continuous
- Lipschitz $⇒$ uniformly continuous

Every $C^1$ function is locally Lipschitz. (Recall that a function $f : \mathbb{R}^m → \mathbb{R}^n$ is said to be $C^1$ if all its first partial derivatives exist and are continuous.)

Definition 10 $^3$ Let $(X, d)$ and $(Y, ρ)$ be metric spaces. A function $f : X → Y$ is called a homeomorphism if it is one-to-one, onto, continuous, and its inverse function is continuous.

Now suppose that $f$ is a homeomorphism and $U ⊂ X$. Let $g : Y → X$ be the inverse of $f$, so $g ∘ f : X → X$ is the identity on $X$, and $f ∘ g : Y → Y$ is the identity on $Y$.

$$y ∈ g^{-1}(U) ⇔ g(y) = f^{-1}(y) ∈ U$$

$$⇔ y ∈ f(U)$$

$U$ open in $X$ $⇒$ $g^{-1}(U)$ is open in $(f(X), ρ)$

$⇒ f(U)$ is open in $(f(X), ρ)$

This says that $(X, d)$ and $(f(X), ρ|_{f(X)})$ are identical in terms of properties that can be characterized solely in terms of open sets; such properties are called “topological properties.”

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$^3$This is the standard definition; de la Fuente instead omits the requirement that $f$ be onto, and requires that $f^{-1}$ be continuous on $f(X)$. See the Corrections handout for a correction to Theorem 6.21
Figure 1: $A$ is open: for every $x \in A$ there is some $\varepsilon > 0$ such that $B_{\varepsilon}(x) \subseteq A$. $B$ is not open: for $x$ depicted in the picture $\exists \varepsilon > 0$ such that $B_{\varepsilon}(x) \subseteq B$. 
Figure 2: Sequences and closed sets
Figure 3: Sequences and closed sets
Figure 4: Proof of Theorem 6.
Figure 5: Proof of Theorem 6.
Figure 6: $f(x) = \frac{1}{x}$ is not uniformly continuous.