Section 3.3. Quotient Vector Spaces

Given a vector space $X$ over a field $F$ and a vector subspace $W$ of $X$, define an equivalence relation by

$$x \sim y \iff x - y \in W$$

Form a new vector space $X/W$: the set of elements of $X/W$ is

$$\{ [x] : x \in X \}$$

where $[x]$ denotes the equivalence class of $x$ with respect to $\sim$. $X/W$ is read “$X$ mod $W$”. Note that the vectors in $X/W$ are sets of vectors in $X$: for $x \in X$,

$$[x] = \{ x + w : w \in W \}$$

We claim that $X/W$ can be viewed as a vector space over $F$. Define the vector space operations $+ , \cdot$ in $X/W$ as follows:

$$[x] + [y] = [x + y]$$
$$\alpha[x] = [\alpha x]$$

The exercise below asks you to verify that these operations are well-defined. Then $X/W$ is a vector space over $F$ with these definitions for $+$ and $\cdot$.

Exercise: Verify that $\sim$ above is an equivalence relation and that vector addition and scalar multiplication are well-defined, i.e.

$$[x] = [x'], [y] = [y'] \Rightarrow [x + y] = [x' + y']$$
$$[x] = [x'], \alpha \in F \Rightarrow [\alpha x] = [\alpha x']$$

Example: Let $X = \mathbb{R}^3$ and let $W = \{ x \in \mathbb{R}^3 : x_1 = x_2 = 0 \}$. Then for $x, y \in \mathbb{R}^3$,

$$x \sim y \iff x - y \in W$$
$$\iff x_1 - y_1 = 0, x_2 - y_2 = 0$$
$$\iff x_1 = y_1, x_2 = y_2$$

and

$$[x] = \{ x + w : w \in W \} = \{ (x_1, x_2, z) : z \in \mathbb{R} \}$$

So the equivalence class corresponding to $x$ is the line in $\mathbb{R}^3$ through $x$ parallel to the axis of the third coordinate. See Figure 1. What is $X/W$? Intuitively this equivalence relation ignores the third coordinate, and we can identify the equivalence class $[x]$ with the vector $(x_1, x_2) \in \mathbb{R}^2$. The next two results show how to formalize this connection.

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1The first part of this material is not in de la Fuente.
Theorem 1 If $X$ is a vector space with $\dim X = n$ for some $n \in \mathbb{N}$ and $W$ is a vector subspace of $X$, then

$$\dim(X/W) = \dim X - \dim W$$

Proof: (Sketch) Begin with a basis $\{w_1, \ldots, w_c\}$ for $W$, and a basis $\{[x_1], \ldots, [x_k]\}$ for $X/W$. Show that

$$\{w_1, \ldots, w_c\} \cup \{x_1, \ldots, x_k\}$$

is a basis for $X$. ■

Theorem 2 Let $X$ and $Y$ be vector spaces over the same field $F$ and $T \in L(X, Y)$. Then $\text{Im} T$ is isomorphic to $X/\ker T$.

Proof: Notice that if $X$ is finite-dimensional, then

$$\dim(X/\ker T) = \dim X - \dim \ker T \quad \text{(by the previous theorem)}$$
$$= \text{Rank} T \quad \text{(by the Rank-Nullity Theorem)}$$
$$= \dim \text{Im} T$$

so $X/\ker T$ is isomorphic to $\text{Im} T$.

We prove that this is true in general, and that the isomorphism is natural.

Define $\tilde{T} : X/\ker T \to \text{Im} T$ by

$$\tilde{T}([x]) = T(x)$$

We first need to check that this is well-defined, that is, that if $[x] = [x']$ then $\tilde{T}([x]) = \tilde{T}([x'])$.

$$[x] = [x'] \Rightarrow x \sim x'$$
$$\Rightarrow x - x' \in \ker T$$
$$\Rightarrow T(x - x') = 0$$
$$\Rightarrow T(x) = T(x')$$

so $\tilde{T}$ is well-defined.

Clearly, $\tilde{T} : X/\ker T \to \text{Im} T$. It is easy to check that $\tilde{T}$ is linear, so $\tilde{T} \in L(X/\ker T, \text{Im} T)$. Next we show that $\tilde{T}$ is an isomorphism.

$$\tilde{T}([x]) = \tilde{T}([y]) \Rightarrow T(x) = T(y)$$
$$\Rightarrow T(x - y) = 0$$
$$\Rightarrow x - y \in \ker T$$
$$\Rightarrow x \sim y$$
$$\Rightarrow [x] = [y]$$
so $\tilde{T}$ is one-to-one.

$$y \in \text{Im } T \implies \exists x \in X \text{ s.t. } T(x) = y$$

$$\implies \tilde{T}([x]) = y$$

so $\tilde{T}$ is onto, hence $\tilde{T}$ is an isomorphism. ■

**Example:** Consider $T \in L(\mathbb{R}^3, \mathbb{R}^2)$ defined by

$$T(x_1, x_2, x_3) = (x_1, x_2)$$

Then $\ker T = \{x \in \mathbb{R}^3 : x_1 = x_2 = 0\}$ is the $x_3$-axis. (Also notice $\ker T = W$ from the previous example.)

Given $x$, the equivalence class $[(x_1, x_2, x_3)]$ is just the line through $x$ parallel to the $x_3$-axis. $\tilde{T}([x]) = T(x_1, x_2, x_3) = (x_1, x_2)$. 

$$\text{Im } T = \mathbb{R}^2, \quad X/\ker T \cong \mathbb{R}^2 = \text{Im } T$$

as we suggested intuitively above (here the symbol $\cong$ denotes isomorphism, that is, we write $Y \cong Z$ if $Y$ and $Z$ are isomorphic.)

Every real vector space $X$ with dimension $n$ is isomorphic to $\mathbb{R}^n$. What’s the isomorphism?

Let $X$ be a finite-dimensional vector space over $\mathbb{R}$ with $\dim X = n$. Fix any Hamel basis $V = \{v_1, \ldots, v_n\}$ of $X$. Any $x \in X$ has a unique representation

$$x = \sum_{j=1}^{n} \beta_j v_j$$

(here, we allow $\beta_j = 0$). (Generally, vectors are represented as column vectors, not row vectors.) Then given the representation of $x$ above, we write

$$\text{crd}_V(x) = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \in \mathbb{R}^n$$

That is, $\text{crd}_V(x)$ is the vector of coordinates of $x$ with respect to the basis $V$.

$$\text{crd}_V(v_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \text{crd}_V(v_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{crd}_V(v_n) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\text{crd}_V$ is an isomorphism from $X$ to $\mathbb{R}^n$. 3
Matrix Representation of a Linear Transformation

Suppose $T \in L(X, Y)$, $\dim X = n$ and $\dim Y = m$. Fix bases

\begin{align*}
V &= \{v_1, \ldots, v_n\} \text{ of } X \\
W &= \{w_1, \ldots, w_m\} \text{ of } Y
\end{align*}

$T(v_j) \in Y$, so

$$T(v_j) = \sum_{i=1}^{m} \alpha_{ij} w_i$$

Define

$$M_{tx_{W,V}}(T) = \begin{pmatrix}
\alpha_{11} & \cdots & \alpha_{1n} \\
\vdots & \ddots & \vdots \\
\alpha_{m1} & \cdots & \alpha_{mn}
\end{pmatrix}$$

Notice that the columns are the coordinates (expressed with respect to $W$) of $T(v_1), \ldots, T(v_n)$.

Observe

$$\begin{pmatrix}
\alpha_{11} & \cdots & \alpha_{1n} \\
\vdots & \ddots & \vdots \\
\alpha_{m1} & \cdots & \alpha_{mn}
\end{pmatrix} \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
\alpha_{11} \\
\vdots \\
\alpha_{m1}
\end{pmatrix}$$

so

$$M_{tx_{W,V}}(T) \cdot \text{crd}_V(v_j) = \text{crd}_W(T(v_j))$$

$$M_{tx_{W,V}}(T) \cdot \text{crd}_V(x) = \text{crd}_W(T(x)) \quad \forall x \in X$$

Multiplying a vector by a matrix does two things:

- Computes the action of $T$
- Accounts for the change in basis

Example: $X = Y = \mathbb{R}^2$, $V = \{(1,0),(0,1)\}$, $W = \{(1,1),(-1,1)\}$, $T = \text{id}$, that is, $T(x) = x$ for all $x$.

$$M_{tx_{W,V}}(T) \neq \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}$$

$M_{tx_{W,V}}(T)$ is the matrix that changes basis from $V$ to $W$. How do we compute it?

$$v_1 = (1,0) = \alpha_{11}(1,1) + \alpha_{21}(-1,1)$$

$$\alpha_{11} - \alpha_{21} = 1$$

$$\alpha_{11} + \alpha_{21} = 0$$

$$2\alpha_{11} = 1, \quad \alpha_{11} = \frac{1}{2}$$
\[ \alpha_{21} = -\frac{1}{2} \]
\[ v_2 = (0, 1) = \alpha_{12}(1, 1) + \alpha_{22}(-1, 1) \]
\[ \alpha_{12} - \alpha_{22} = 0 \]
\[ \alpha_{12} + \alpha_{22} = 1 \]
\[ 2\alpha_{12} = 1, \alpha_{12} = \frac{1}{2} \]
\[ \alpha_{22} = \frac{1}{2} \]
\[ Mtx_{W,V}(id) = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \]

Theorem 3 (Thm. 3.5') Let \( X \) and \( Y \) be vector spaces over the same field \( F \), with \( \dim X = n \), \( \dim Y = m \). Then \( L(X, Y) \), the space of linear transformations from \( X \) to \( Y \), is isomorphic to \( F_{m \times n} \), the vector space of \( m \times n \) matrices over \( F \). If \( V = \{v_1, \ldots, v_n\} \) is a basis for \( X \) and \( W = \{w_1, \ldots, w_m\} \) is a basis for \( Y \), then
\[ Mtx_{W,V} \in L(L(X, Y), F_{m \times n}) \]
and \( Mtx_{W,V} \) is an isomorphism from \( L(X, Y) \) to \( F_{m \times n} \).

Theorem 4 (From Handout) Let \( X, Y, Z \) be finite-dimensional vector spaces over the same field \( F \) with bases \( U, V, W \) respectively. Let \( S \in L(X, Y) \) and \( T \in L(Y, Z) \). Then
\[ Mtx_{W,V}(T) \cdot Mtx_{V,U}(S) = Mtx_{W,U}(T \circ S) \]
i.e. matrix multiplication corresponds via the matrix representation isomorphism to composition of linear transformations.

Proof: See handout. \( \blacksquare \)

Note that \( Mtx_{W,V} \) is a function from \( L(X, Y) \) to the space \( F_{m \times n} \) of \( m \times n \) matrices, while \( Mtx_{W,V}(T) \) is an \( m \times n \) matrix.

The theorem can be summarized by the following “Commutative Diagram:"

\[
\begin{array}{c c c c c c}
& & S & & T \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \rightarrow & Y & \rightarrow & Z \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
R^n & \rightarrow & R^m & \rightarrow & R^r \\
\downarrow & & \downarrow & & \downarrow \\
Mtx_{V,U}(S) & & Mtx_{W,V}(T)
\end{array}
\]

We say the diagram commutes because you get the same answer any way you go around the diagram (in directions allowed by the arrows). The \( \text{crd} \) arrows go in both directions because \( \text{crd} \) is an isomorphism.
Section 3.5. Change of Basis and Similarity

Let $X$ be a finite-dimensional vector space with basis $V$. If $T \in L(X, X)$ it is customary to use the same basis in the domain and range. In this case,

$$Mtx_V(T)$$

denotes $Mtx_{V,V}(T)$

Question: If $W$ is another basis for $X$, how are $Mtx_V(T)$ and $Mtx_W(T)$ related?

$$Mtx_{V,W}(id) \cdot Mtx_W(T) \cdot Mtx_{W,V}(id) = Mtx_{V,W}(id) \cdot Mtx_{W,V}(T \circ id) = Mtx_{V,V}(id \circ T \circ id) = Mtx_V(T)$$

and

$$Mtx_{V,W}(id) \cdot Mtx_{W,V}(id) = Mtx_{V,V}(id) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

So this says that

$$Mtx_V(T) = P^{-1}Mtx_W(T)P$$

for the invertible matrix

$$P = Mtx_{W,V}(id)$$

that is the change of basis matrix. On the other hand, if $P$ is any invertible matrix, then $P$ is also a change of basis matrix for appropriate corresponding bases (see handout).

Definition 5 Square matrices $A$ and $B$ are similar if

$$A = P^{-1}BP$$

for some invertible matrix $P$.

Theorem 6 Suppose that $X$ is a finite-dimensional vector space.

1. If $T \in L(X, X)$ then any two matrix representations of $T$ are similar. That is, if $U, W$ are any two bases of $X$, then $Mtx_W(T)$ and $Mtx_U(T)$ are similar.

2. Conversely, two similar matrices represent the same linear transformation $T$, relative to suitable bases. That is, given similar matrices $A, B$ with $A = P^{-1}BP$ and any basis $U$, there is a basis $W$ and $T \in L(X, X)$ such that

$$B = Mtx_U(T)$$

$$A = Mtx_W(T)$$

$$P = Mtx_{U,W}(id)$$

$$P^{-1} = Mtx_{W,U}(id)$$
Proof: See Handout on Diagonalization and Quadratic Forms.

Section 3.6. Eigenvalues and Eigenvectors

Here, we define eigenvalues and eigenvectors of a linear transformation and show that $\lambda$ is an eigenvalue of $T$ if and only if $\lambda$ is an eigenvalue for some matrix representation of $T$ if and only if $\lambda$ is an eigenvalue for every matrix representation of $T$.

Definition 7 Let $X$ be a vector space and $T \in L(X, X)$. We say that $\lambda$ is an eigenvalue of $T$ and $v \neq 0$ is an eigenvector corresponding to $\lambda$ if $T(v) = \lambda v$.

Theorem 8 (Theorem 4 in Handout) Let $X$ be a finite-dimensional vector space, and $U$ a basis. Then $\lambda$ is an eigenvalue of $T$ if and only if $\lambda$ is an eigenvalue of $Mtx_U(T)$. $v$ is an eigenvector of $T$ corresponding to $\lambda$ if and only if $crd_U(v)$ is an eigenvector of $Mtx_U(T)$ corresponding to $\lambda$.

Proof: By the Commutative Diagram Theorem,

$T(v) = \lambda v \iff crd_U(T(v)) = crd_U(\lambda v) \iff Mtx_U(T)(crd_U(v)) = \lambda (crd_U(v))$

Computing eigenvalues and eigenvectors:

Suppose dim $X = n$; let $I$ be the $n \times n$ identity matrix. Given $T \in L(X, X)$, fix a basis $U$ and let

$A = Mtx_U(T)$

Find the eigenvalues of $T$ by computing the eigenvalues of $A$:

$Av = \lambda v \iff (A - \lambda I)v = 0 \iff (A - \lambda I)$ is not invertible $\iff \det(A - \lambda I) = 0$
We have the following facts:

- If $A \in \mathbb{R}^{n \times n}$,
  
  $$f(\lambda) = \det(A - \lambda I)$$

  is an $n^{th}$ degree polynomial in $\lambda$ with real coefficients; it is called the characteristic polynomial of $A$.

- $f$ has $n$ roots in $\mathbb{C}$, counting multiplicity:
  
  $$f(\lambda) = (\lambda - c_1)(\lambda - c_2) \cdots (\lambda - c_n)$$

  where $c_1, \ldots, c_n \in \mathbb{C}$ are the eigenvalues; the $c_j$'s are not necessarily distinct. Notice that $f(\lambda) = 0$ if and only if $\lambda \in \{c_1, \ldots, c_n\}$, so the roots are the solutions of the equation $f(\lambda) = 0$.

- the roots that are not real come in conjugate pairs:
  
  $$f(a + bi) = 0 \iff f(a - bi) = 0$$

- if $\lambda = c_j \in \mathbb{R}$, there is a corresponding eigenvector in $\mathbb{R}^n$.

- if $\lambda = c_j \not\in \mathbb{R}$, the corresponding eigenvectors are in $\mathbb{C}^n \setminus \mathbb{R}^n$.

### Diagonalization

**Definition 9** Suppose $X$ is a finite-dimensional vector space with basis $U$. Given a linear transformation $T \in L(X, X)$, let

$$A = M_{txU}(T)$$

We say that $A$ can be diagonalized (or is diagonalizable) if there is a basis $W$ for $X$ such that $M_{txW}(T)$ is diagonal, i.e.

$$M_{txW}(T) = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

Notice that the eigenvectors of $M_{txW}(T)$ are exactly the standard basis vectors of $\mathbb{R}^n$. But $w_j$ is an eigenvector of $T$ corresponding to $\lambda_j$ if and only if $crd_W(w_j)$ is an eigenvector of $M_{txW}(T)$, and $crd_W(w_j)$ is the $j^{th}$ standard basis vector of $\mathbb{R}^n$, so $W = \{w_1, \ldots, w_n\}$ where $w_j$ is an eigenvector corresponding to $\lambda_j$.

Then the action of $T$ is clear: it stretches each basis element $w_i$ by the factor $\lambda_i$. 

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Theorem 10 (Thm. 6.7') Let $X$ be an $n$-dimensional vector space, $T \in L(X, X)$, $U$ any basis of $X$, and $A = Mtx_U(T)$. Then the following are equivalent:

1. $A$ can be diagonalized
2. there is a basis $W$ for $X$ consisting of eigenvectors of $T$
3. there is a basis $V$ for $\mathbb{R}^n$ consisting of eigenvectors of $A$

Proof: Follows from Theorem 6.7 in de la Fuente and Theorem 4 from the Handout.

Theorem 11 (Thm. 6.8’) Let $X$ be a vector space and $T \in L(X, X)$.

1. If $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of $T$ with corresponding eigenvectors $v_1, \ldots, v_m$, then \{\(v_1, \ldots, v_m\)\} is linearly independent.
2. If $\dim X = n$ and $T$ has $n$ distinct eigenvalues, then $X$ has a basis consisting of eigenvectors of $T$; consequently, if $U$ is any basis of $X$, then $Mtx_U(T)$ is diagonalizable.

Proof: This is an adaptation of the proof of Theorem 6.8 in de la Fuente.
Figure 1: An illustration of $X/W$ where $X = \mathbb{R}^3$ and $W = \{x \in \mathbb{R}^3 : x_1 = x_2 = 0\}$. Here $[x] = \{(x_1, x_2, z) : z \in \mathbb{R}\}$ is the line through $x$ parallel to the axis of the third coordinate.