Econ 204 – Problem Set 1

Due Friday, July 29

1. Certain subsets of a given set S are called A-sets and others are called B-sets. Suppose that these subsets are chosen in such a way that the following properties are satisfied:

   - The union of any collection of A-sets is and A-set.
   - The intersection of any finite number of A-sets is an A-set.
   - The complement of an A-set is a B-set and the complement of a B-set is an A-set.

   Prove the following:

   i. The intersection of any collection of B-sets is a B-set.

   ii. The union of any finite number of B-sets is a B-set.

2. Use induction to prove the following:

   (a) $1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n}$

   (b) Let F_k be the Fibonacci numbers. Show that

   $$F_{n-1}F_{n+1} = F_n^2 + (-1)^n, \ \sum_{i=0}^{n} F_i^2 = F_n F_{n+1}$$

   (c) Given a sequence $X = \{x_1, \ldots, x_n\}$ of positive numbers, define the arithmetic, geometric, and harmonic means as follows:

   $$A(X) = \frac{x_1 + \ldots + x_n}{n}$$
   $$G(X) = \sqrt[n]{x_1 \cdots x_n}$$
   $$H(X) = \frac{n}{\frac{1}{x_1} + \ldots + \frac{1}{x_n}}$$

   Prove that $H(X) \leq G(X) \leq A(X)$.

3. In the following examples, show that the sets A and B are numerically equivalent by finding a specific bijection between the two.

   (a) $A = [0, 1], \ B = [10, 20]$

   (b) $A = [0, 1], \ B = [0, 1)$

   (c) $A = (0, 1), \ B = (0, \infty)$

   (d) $A = (0, 1), \ B = \mathbb{R}$

4. Show that if $(A \setminus B)$ and $(B \setminus A)$ are numerically equivalent, then so are A and B.

5. Let $f : A \to A$. Prove that there is a unique largest subset $X \subset A$ such that $f(X) = X$ (here if $Y \subset X$ we call X "larger" than Y).

6. Suppose a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ has the following properties for all x, y:

   - $f(x) = 0 \iff x = 0$
   - $x \geq y \implies f(x) \geq f(y)$
   - $f(x + y) \leq f(x) + f(y)$

   Show that if (X, d) is a metric space, then $(X, f \circ d)$ is also a metric space.