Econ 204 – Problem Set 2

Due Tuesday, August 2

1. Prove that the set of cluster points of a sequence $\{x_k\}$ is closed.

2. Let X be an arbitrary set, and $A, B \subset X$, such that $\text{int}A = \text{int}B = \emptyset$.

   (a) Prove that if A is closed in X, then $\text{int}(A \cup B) = \emptyset$

   (b) If A is not necessarily closed the previous statement is not true. Give an example where $\text{int}(A \cup B) \neq \emptyset$ in this case.

3. Consider l^∞, the set of bounded sequences (so $a \in l^\infty$ if and only if $a = (a_1, a_2, \ldots)$ with $a_i \in \mathbb{R}$ for all i and there exists some $M < \infty$ such that $|a_i| \leq M$ for all i).

   (a) Show that l^∞ is a vector space over $\mathbb{R}$.

   (b) Show that the function $f(a) = \sup_i |a_i|$ defines a norm on this vector space.

   (c) Consider the subspace l^0 of sequences with only a finite number of nonzero elements (so $a = (a_1, a_2, \ldots) \in l^0$ if and only if there exists some $M < \infty$ such that $a_i = 0$ for all $i \geq M$). Does this define a closed subspace of l^∞?

4. Prove that any bounded sequence in $\mathbb{R}^2$ has a convergent subsequence!

5. Let (X, d) be the space of continuous real-valued functions on a closed interval $[0, b]$ with the metric induced by the sup norm, ie. $X = C([0, b])$ and $d(f, g) = \max_x |f(x) - g(x)|$, for some $b < 1$. Let be $T : X \to X$:

$$(Tf)(x) = \int_0^x f(u)du + g(t),$$

   for some continuous function $g : [0, b] \to \mathbb{R}$. Prove that T has a unique fixed point. (Note that for the Contraction Mapping Theorem, you need to show that (X, d) is a complete metric space. You might want to use that $\mathbb{R}$ is complete.)