Econ 204 – Problem Set 3

Due Friday, August 5

1. A function $f : X \to Y$ is *open* if for every open set $A \subset X$, its image $f(A)$ is also open. Show that any continuous open function from $\mathbb{R}$ into $\mathbb{R}$ (with the usual metric) is strictly monotonic.

2. Take any mapping f from a metric space X into a metric space Y. Prove that f is continuous if and only if $f\left(\overline{A}\right) \subseteq \overline{f(A)}$ for every set A. (Hint: use the closed set characterization of continuity).

3. Show that in a metric space, a set is closed if and only its intersection with any compact set is closed.

4. Let (X, d) and (Y, ρ) be metric spaces, where (X, d) is complete. Suppose $C \subset X$ is closed, and the continuous function $f : C \to Y$ satisfies the following:
$$\rho\left(f(a), f(b)\right) \geq d(a, b)$$
for all $a, b \in C$. Prove that $f(C)$ is closed.

5. Show that a metric space X is connected if and only if every continuous function $f : X \to \{0, 1\}$ is constant.

6. For any correspondence $\Gamma : X \to 2^Y$, define the following:
   - Γ is *injective* if $\Gamma(x) \cap \Gamma(x') = \varnothing$ whenever $x \neq x'$.
   - Γ is *surjective* if $\Gamma(X) = Y$.
   - Γ is *bijective* if it is both injective and surjective.

   Show that Γ is bijective if and only if $\Gamma = f^{-1}$ where f^{-1} is the pre-image of some function $f : Y \to X$.

7. Define the correspondence $\Gamma : [0, 1] \to 2^{[0,1]}$ by:
$$\Gamma(x) = \begin{cases} [0,1] \cap \mathbb{Q} & \text{if } x \in [0,1] \setminus \mathbb{Q} \\ [0,1] \setminus \mathbb{Q} & \text{if } x \in [0,1] \cap \mathbb{Q} \end{cases}$$
Show that Γ (i) is nowhere upper hemicontinuous; (ii) does not have a closed graph; (iii) is everywhere lower hemicontinuous.