Econ 204 – Problem Set 3
Due Friday, August 5

1. A function $f : X \to Y$ is open if for every open set $A \subset X$, its image $f(A)$ is also open. Show that any continuous open function from $\mathbb{R}$ into $\mathbb{R}$ (with the usual metric) is strictly monotonic.

2. Take any mapping $f$ from a metric space $X$ into a metric space $Y$. Prove that $f$ is continuous if and only if $f(A) \subseteq f(A)$ for every set $A$. (Hint: use the closed set characterization of continuity).

3. Show that in a metric space, a set is closed if and only its intersection with any compact set is closed.

4. Let $(X, d)$ and $(Y, \rho)$ be metric spaces, where $(X, d)$ is complete. Suppose $C \subset X$ is closed, and the continuous function $f : C \to Y$ satisfies the following:

\[ \rho(f(a), f(b)) \geq d(a, b) \]

for all $a, b \in C$. Prove that $f(C)$ is closed.

5. Show that a metric space $X$ is connected if and only if every continuous function $f : X \to \{0, 1\}$ is constant.

6. For any correspondence $\Gamma : X \to 2^Y$, define the following:

- $\Gamma$ is injective if $\Gamma(x) \cap \Gamma(x') = \emptyset$ whenever $x \neq x'$.
- $\Gamma$ is surjective if $\Gamma(X) = Y$.
- $\Gamma$ is bijective if it is both injective and surjective.

Show that $\Gamma$ is bijective if and only if $\Gamma = f^{-1}$ where $f^{-1}$ is the pre-image of some function $f : Y \to X$.

7. Define the correspondence $\Gamma : [0, 1] \to 2^{[0, 1]}$ by:

\[
\Gamma(x) = \begin{cases} 
[0, 1] \cap \mathbb{Q} & \text{if } x \in [0, 1] \setminus \mathbb{Q} \\
[0, 1] \setminus \mathbb{Q} & \text{if } x \in [0, 1] \cap \mathbb{Q}
\end{cases}
\]

Show that $\Gamma$ (i) is nowhere upper hemicontinuous; (ii) does not have a closed graph; (iii) is everywhere lower hemicontinuous.