Econ 204 – Problem Set 4

Due Tuesday, August 9.

1. Determine whether or not each of the following sets is a vector space. In case it is, find the dimension of the space and a Hamel basis for it.

   (a) The set of solutions in $\mathbb{R}^3$ to the following system of linear equations, with vector addition and scalar multiplication defined in the usual way

   $$\begin{aligned} x_1 - 5x_2 + 2x_3 &= 0 \\ 5x_1 + 3x_2 - x_3 &= 0. \end{aligned}$$

   (b) The set of $n \times n$ matrices having a trace equal to one, with matrix addition and scalar multiplication defined in the usual way.

   (c) The set of $m \times n$ matrices having all their elements sum-up to zero, with matrix addition and scalar multiplication defined in a usual way.

   (d) The set of 2×1 matrices with real elements, with addition and scalar multiplication defined as

   $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \end{pmatrix}$$

   $$\alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix}$$

   (e) All strictly positive reals $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$ with vector addition defined as $x + y = xy$ using the multiplication in the one dimensional Euclidian space and scalar multiplication defined as $\alpha x = x^\alpha$.

2. A linear transformation $T : \mathbb{R}^k, \to \mathbb{R}^k$ has the property that $T^n = 0$ for some integer $n > 0$.

   (a) Show that T is not invertible.

   (b) Show that $T + I$ is invertible. (Hint: the inverse is a polynomial function of T)

3. Prove that any set of pairwise orthogonal nonzero vectors is linearly independent.

4. (a) Let T be clockwise rotation by 90 and halving in length in $\mathbb{R}^2$. Find the matrix representation $Mtx_{W,V}(T)$ of T given bases:

   i. $W = V = \{(1,0),(0,1)\}$.
   ii. $W = V = \{(1,1),(1,-1)\}$.
   iii. $W = \{(1,0),(0,1)\}, V = \{(1,1),(1,-1)\}$.
   iv. $W = \{(1,1),(1,-1)\}, V = \{(1,0),(0,1)\}$.

(b) Let T be projection on to the horizontal axis in $\mathbb{R}^2$. Find the matrix representation $Mtx_{W,V}(T)$ of T given bases:

   i. $W = V = \{(1,0),(0,1)\}$.
   ii. $W = \{(1,1),(1,-1)\}, V = \{(1,0),(0,1)\}$.

(c) Are either of the above transformations injective? surjective? What are their kernels and ranges? (One line answers are fine, no detailed proofs required.)

Solution

5. Suppose that V is finite dimensional and that $T \in L(V,W)$ (W is not necessarily finite-dimensional). Prove that there exists a subspace U of V such that $\mathrm{Ker}(T) \cap U = \{0\}$ and that $\mathrm{Im}(T) = \{Tu : u \in U\}$.