Econ 204 – Problem Set 5

Due Friday, August 12

1. Suppose $f : [a, b] \to \mathbb{R}$ is a differentiable function. If $f(a) = 0$ and $|f'(x)| \leq A|f(x)|$ for some real number A, show that $f(x) = 0$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Prove that $f'(\mathbb{R})$, the image of the derivative function, is an interval (possibly a singleton).

3. If $a_0 + \frac{1}{2}a_1 + \cdots + \frac{1}{n}a_{n-1} + \frac{1}{n+1}a_n = 0$, where $a_0, \ldots, a_n$ are real constants, prove that the equation

$$a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n = 0$$

has at least one real root between 0 and 1.

4. Assume that $f : [0, \infty) \to \mathbb{R}$ is differentiable for all $x > 0$, and $f'(x) \to 0$ as $x \to \infty$. Prove

$$\lim_{x \to \infty} [f(x + 1) - f(x)] \to 0.$$

5. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x, y) = (e^y \cos(x), e^y \sin(x))$.[1]

   (i) Show that F satisfies the prerequisites of the Inverse Function Theorem for all $(x, y) \in \mathbb{R}^2$ (and is therefore locally injective everywhere) but F is not globally injective.

   (ii) Compute the Jacobian of the local inverse of F and evaluate it at $F\left(\frac{\pi}{3}, 0\right)$.

   (iii) Find an explicit formula for the continuous inverse of F mapping a neighborhood of $F\left(\frac{\pi}{3}, 0\right)$ into a neighborhood of $\left(\frac{\pi}{3}, 0\right)$ and verify that its Jacobian at $F(\frac{\pi}{3}, 0)$ equals the one you calculated in (ii).

[1] In this problem, take the standard trigonmetric identities as given.