1. Consider the function \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) given by
\[
f(x_1, x_2) = (x_1^2 + x_2 + 1, x_1x_2)
\]

(a) At which points can we apply the inverse function theorem?

(b) Let \( x = (x_1, x_2) \) be one of the points you found in (a). We know from the Inverse Function Theorem that in some neighborhood of \( x \), \( f \) has an inverse. What is the derivative of that inverse at \( f(x) \)?

2. Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a \( C^1 \) function. Define \( F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) by
\[
F(x, \omega) = f(x) + \omega
\]
Show that there is a set \( \Omega_0 \subset \mathbb{R}^n \) of Lebesgue measure zero such that if \( \omega \not\in \Omega_0 \) then for each \( x_0 \) satisfying \( F(x_0, \omega_0) = 0 \) there is an open set \( U \) containing \( x_0 \), an open set \( V \) containing \( \omega_0 \), and a \( C^1 \) function \( h : V \to U \) such that for all \( \omega \in V \), \( x = h(\omega) \) is the unique element of \( U \) satisfying \( F(x, \omega) = 0 \).

3. Suppose \( \Gamma : X \to 2^Y \) is a correspondence defined by \( \Gamma(x) = \{ f_1(x), \ldots, f_N(x) \} \) where \( f_i : X \to Y \) is a continuous function for each \( i \in \{1, \ldots, N\} \). Prove that \( \Gamma \) is both uhc and lhc.

4. Let \( A \) be a nonempty, compact and convex subset of \( \mathbb{R}^2 \) such that if \( (x, y) \in A \) for some \( x, y \in \mathbb{R} \) then there exists some \( z \in \mathbb{R} \) such that \( (y, z) \in A \). Prove that \( (x^*, x^*) \in A \) for some \( x^* \in \mathbb{R} \). (Hint: Use Kakutani’s Fixed Point Theorem.)

5. Let \( A \) and \( B \) be nonempty, convex subsets of \( \mathbb{R}^n \) with \( \text{int } A \neq \emptyset \). Using the Separating Hyperplane Theorem, prove that there exists \( p \in \mathbb{R}^n \)
with \( p \neq 0 \) such that \( \sup p \cdot A \leq \inf p \cdot B \) if and only if \( \text{int} A \cap B = \emptyset \).

(Hint: You might want to use the result of Theorem 1.11 in de la Fuente p. 23.)

6. Consider the second order linear differential equation given by

\[
y'' = -y - y'
\]

(a) Show how this equation can be rewritten as the following first order linear differential equation of two variables:

\[
\bar{x}'(t) = A\bar{x}(t),
\]

where \( A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \) and \( \bar{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \).

(b) Describe the solutions of the first order system (verbally) by analyzing the matrix \( A \).

(c) In a phase diagram, show the behavior of the system using the previous analysis and by solving for \( x_1'(t) = 0 \) and \( x_2'(t) = 0 \).

(d) Give the solution of the system when \( x_1(t_0) = 0 \) and \( x_2(t_0) = 1 \).