Econ 204 – Problem Set 6

Due 9am Monday, August 15 in Tamas mailbox.

1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x_1, x_2) = (x_1^2 + x_2 + 1, x_1 x_2)$$

   (a) At which points can we apply the inverse function theorem?

   (b) Let $x = (x_1, x_2)$ be one of the points you found in (a). We know from the Inverse Function Theorem that in some neighborhood of x, f has an inverse. What is the derivative of that inverse at $f(x)$?

2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function. Define $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$F(x, \omega) = f(x) + \omega$$

   Show that there is a set $\Omega_0 \subset \mathbb{R}^n$ of Lebesgue measure zero such that if $\omega \notin \Omega_0$ then for each x_0 satisfying $F(x_0, \omega_0) = 0$ there is an open set U containing x_0, an open set V containing ω_0, and a C^1 function $h : V \to U$ such that for all $\omega \in V$, $x = h(\omega)$ is the unique element of U satisfying $F(x, \omega) = 0$.

3. Suppose $\Gamma : X \to 2^Y$ is a correspondence defined by $\Gamma(x) = \{f_1(x), \ldots, f_N(x)\}$ where $f_i : X \to Y$ is a continuous function for each $i \in \{1, \ldots, N\}$. Prove that Γ is both uhc and lhc.

4. Let A be a nonempty, compact and convex subset of $\mathbb{R}^2$ such that if $(x, y) \in A$ for some $x, y \in \mathbb{R}$ then there exists some $z \in \mathbb{R}$ such that $(y, z) \in A$. Prove that $(x^*, x^*) \in A$ for some $x^* \in \mathbb{R}$. (Hint: Use Kakutani's Fixed Point Theorem.)

5. Let A and B be nonempty, convex subsets of $\mathbb{R}^n$ with $\operatorname{int} A \neq \emptyset$. Using the Separating Hyperplane Theorem, prove that there exists $p \in \mathbb{R}^n$

with $p \neq 0$ such that $\sup\ p \cdot A \leq \inf\ p \cdot B$ if and only if $\text{int}\ A \cap B = \emptyset$. (Hint: You might want to use the result of Theorem 1.11 in de la Fuente p. 234.)

6. Consider the second order linear differential equation given by

$$y'' = -y - y'$$

   (a) Show how this equation can be rewritten as the following *first* order linear differential equation of two variables:

$$\bar{x}'(t) = A\bar{x}(t),$$

   where $A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ and $\bar{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

   (b) Describe the solutions of the first order system (verbally) by analyzing the matrix A.

   (c) In a phase diagram, show the behavior of the system using the previous analysis and by solving for $x_1'(t) = 0$ and $x_2'(t) = 0$.

   (d) Give the solution of the system when $x_1(t_0) = 0$ and $x_2(t_0) = 1$.