Economics 204 Summer/Fall 2016
Final Exam

Answer all of the questions below. Be as complete, correct, and concise as possible. There are 6 questions for a total of 165 points possible; point values for each problem are in parentheses. For questions with subparts, each subpart is worth the same number of points. You have 180 minutes to complete the exam. Use the points as a guide to allocating your time. You may use any result from class with appropriate references unless you are specifically being asked to prove it.

Do not turn the page until the exam begins.

1. (15) Define or state each of the following.

   (a) open set in a metric space (X, d)

   (b) basis for a vector space X

   (c) Brouwer's Fixed Point Theorem

2. (30) Let k be a positive integer. Prove that for every $n \in \mathbf{N}_0 = \{0, 1, 2, \ldots\}$,

$$(k^2 + n)! \geq k^{2n}$$

3. (30) Let $f : [a, b] \to \mathbf{R}$ where $a, b \in \mathbf{R}$ with $a < b$. Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Suppose f is monotonically increasing on $[a, b]$. Show that $f'(x) \geq 0$ for all $x \in (a, b)$.

4. (30) Let X and Y be vector spaces over the same field F, with $\dim X = \dim Y = n$ for some $n \in \mathbf{N}$. Let $T : X \to Y$ be a linear transformation.

   (a) Show that $\operatorname{Im} T$ is a vector subspace of Y.

   (b) Show that T is invertible if and only if $\ker T = \{0\}$.

5. (30) Let X and Y be normed vector spaces. Let $f : C \to Y$ where $C \subseteq X$ is compact. Suppose that for each $x \in C$ there exists $\delta > 0$ and a continuous function $g_x : C \to \mathbf{R}$ such that for all $z \in B_\delta(x)$,

$$\|f(x) - f(z)\| \leq g_x(z)$$

   Show that f is bounded.

   (**Hint:** Use the open cover defintion of compactness.)

6. (30) Suppose $\Psi : X \to 2^Y$ is a correspondence with nonempty and compact values, where $X \subseteq \mathbf{R}^n, Y \subseteq \mathbf{R}^m$ for some n, m. Suppose there exists $\beta \in (0, 1)$ such that for all $x, y \in X$,

$$\sup\{\|z - w\| : z \in \Psi(x),\ w \in \Psi(y)\} \leq \beta\|x - y\|$$

   Show directly from the definition of upper hemicontinuity that Ψ is upper hemicontinuous.

   (**Note:** For full credit, the answer will have to directly use the definition of upper hemicontinuity. An otherwise correct answer that uses alternative characterizations of upper hemicontinuity will receive 15 points, provided any necessary additional assumptions are clearly stated.)