Econ 204 2017

Lecture 10

Outline

1. Diagonalization of Real Symmetric Matrices
2. Application to Quadratic Forms
3. Linear Maps Between Normed Spaces

Announcements
- PS 3 due now
- Solutions due 2pm today
- PS 4 posted
- Last year’s exam posted on Saturday
How Might This Matter

• Why does diagonalizability matter?

Consider a two-dimensional linear difference equation:

\[
\begin{pmatrix}
  c_{t+1} \\
  k_{t+1}
\end{pmatrix} = \begin{pmatrix}
  b_{11} & b_{12} \\
  b_{21} & b_{22}
\end{pmatrix} \begin{pmatrix}
  c_t \\
  k_t
\end{pmatrix} \quad \forall t = 0, 1, 2, 3, \ldots
\]

given an initial condition \( c_0, k_0 \), or, setting

\[
y_t = \begin{pmatrix}
  c_t \\
  k_t
\end{pmatrix} \quad \forall t \quad \text{and} \quad B = \begin{pmatrix}
  b_{11} & b_{12} \\
  b_{21} & b_{22}
\end{pmatrix}
\]

we can rewrite this more compactly as

\[
y_{t+1} = By_t \quad \forall t
\]

where \( b_{ij} \in \mathbb{R} \) each \( i, j \).
We want to find a solution \( y_t, \ t = 1, 2, 3, \ldots \) given initial condition \( y_0 \). (Why?)

Such a dynamical system will arise for example as a characterization of the solution to a standard infinite-horizon optimal growth problem (202a, lecture 2).

If \( B \) is diagonalizable, this can be easily solved after a change of basis. If \( B \) is diagonalizable, choose an invertible \( 2 \times 2 \) real matrix \( P \) such that

\[
P^{-1}BP = D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}
\]

Then

\[
y_{t+1} = By_t \quad \forall t \iff P^{-1}y_{t+1} = P^{-1}By_t \quad \forall t \iff \begin{pmatrix} P^{-1}y_{t+1} \end{pmatrix} = \begin{pmatrix} P^{-1}BP \end{pmatrix} P^{-1}y_t \quad \forall t \iff \bar{y}_{t+1} = D\bar{y}_t \quad \forall t
\]

where \( \bar{y}_t = P^{-1}y_t \quad \forall t \)
where \( \bar{y}_t = P^{-1}y_t \; \forall t. \)

Since \( D \) is diagonal, after a change of basis to \( \bar{y}_t \), we need to solve two independent linear univariate difference equations, which is easy:

\[
\bar{y}_{it} = d_i^t \bar{y}_{i0} \quad \forall t
\]

- Not all real \( n \times n \) matrices are diagonalizable (not even all invertible \( n \times n \) matrices are)... so can we identify some classes that are? yesterday:  
  - basis of eigenvectors \((\Rightarrow)\)  
  - \( n \) distinct eigenvalues \((\Rightarrow)\)

- Some types of matrices appear more frequently than others – especially real symmetric \( n \times n \) matrices (matrix representation of second derivatives of \( C^2 \) functions, quadratic forms...).

  e.g. second order conditions in optimization, checking concavity and convexity, Taylor series approximation of function
Recall that an $n \times n$ real matrix $A$ is symmetric if $a_{ij} = a_{ji}$ for all $i, j$, where $a_{ij}$ is the $(i, j)^{th}$ entry of $A$. 


Rest of this section: work in $\mathbb{R}^n$
- vector space
- norm
- inner product ($x \cdot y = \sum_{i=1}^{n} x_i y_i$)

Orthonormal Bases

Definition 1. Let

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

A basis $V = \{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ is orthonormal if $v_i \cdot v_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{if } i \neq j \end{cases}$

In other words, a basis is orthonormal if each basis element has unit length ($\|v_i\|^2 = v_i \cdot v_i = 1 \ \forall i$), and distinct basis elements are perpendicular ($v_i \cdot v_j = 0$ for $i \neq j$).
Orthonormal Bases

**Remark**: Suppose that \( x = \sum_{j=1}^{n} \alpha_j v_j \) where \( \{v_1, \ldots, v_n\} \) is an orthonormal basis of \( \mathbb{R}^n \). Then

\[
x \cdot v_k = \left( \sum_{j=1}^{n} \alpha_j v_j \right) \cdot v_k = \sum_{j=1}^{n} \alpha_j (v_j \cdot v_k) = \sum_{j=1}^{n} \alpha_j \delta_{jk} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}
\]

so

\[
x = \sum_{j=1}^{n} (x \cdot v_j) v_j
\]
Orthonormal Bases

Example: The standard basis of $\mathbb{R}^n$ is orthonormal.

$$\mathbf{e}_i = (0, \ldots, 1, 0, \ldots, 0) \quad i = 1, \ldots, n$$

(Why?)

e.g. $\mathbb{R}^2$: $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$

others? e.g. $\mathbf{v}_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{5}})$, $\mathbf{v}_2 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

also, many bases that are not orthonormal
Unitary Matrices

Recall that for a real $n \times m$ matrix $A$, $A^\top$ denotes the transpose of $A$: the $(i, j)^{th}$ entry of $A^\top$ is the $(j, i)^{th}$ entry of $A$.

So the $i^{th}$ row of $A^\top$ is the $i^{th}$ column of $A$.

**Definition 2.** A real $n \times n$ matrix $A$ is unitary if $A^\top = A^{-1}$.

Notice that by definition every unitary matrix is invertible.
Unitary Matrices

**Theorem 1.** A real $n \times n$ matrix $A$ is unitary if and only if the columns of $A$ are orthonormal.

**Proof.** Let $v_j$ denote the $j^{th}$ column of $A$.

\[ A^\top = A^{-1} \iff A^\top A = I = (\delta_{ij}) \iff v_i \cdot v_j = \delta_{ij} \quad \forall i, j \iff \{v_1, \ldots, v_n\} \text{ is orthonormal} \]

\[ \square \]
If \( A \) is unitary, let \( V \) be the set of columns of \( A \) and \( W \) be the standard basis of \( \mathbb{R}^n \). Since \( A \) is unitary, it is invertible, so \( V \) is a basis of \( \mathbb{R}^n \). \( \{v_1, \ldots, v_n\} \) linearly independent.

\[
A^\top = A^{-1} = M_{tx_{V,W}(id)} = \text{change of basis from } W \text{ to } V
\]

Since \( V \) is orthonormal, the transformation between bases \( W \) and \( V \) preserves all geometry, including lengths and angles.
Thus: Let $C$ be an $n \times n$ real symmetric matrix. Then $C$ is diagonalizable. In addition,

$$C = P^{-1}DP$$

where $D$ is a diagonal matrix and $P$ is unitary.

Note: The diagonal elements $d_1, \ldots, d_n$ of $D$ are the eigenvalues of $C$.

- $C$ has orthonormal eigenvectors $v_1, \ldots, v_n$ that are a basis for $\mathbb{R}^n$. 
Diagonalization of Real Symmetric Matrices

**Theorem 2.** Let $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $W$ be the standard basis of $\mathbb{R}^n$. **Suppose that** $\text{Mtx}_W(T)$ **is symmetric.** Then the eigenvectors of $T$ are all real, and there is an orthonormal basis $V = \{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ consisting of eigenvectors of $T$, so that $\text{Mtx}_W(T)$ is diagonalizable:

$$C \quad \text{Mtx}_W(T) = \text{Mtx}_{W,V}(\text{id}) \cdot \text{Mtx}_V(T) \cdot \text{Mtx}_{V,W}(\text{id})$$

where $\text{Mtx}_V T$ is diagonal and the change of basis matrices $\text{Mtx}_{V,W}(\text{id})$ and $\text{Mtx}_{W,V}(\text{id})$ are unitary.

The proof of the theorem requires a lengthy digression into the linear algebra of complex vector spaces. A brief outline is in the notes.
### Quadratic Forms

**Example:** Let

\[ f(x) = \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2 \]

Let write as \( f(x) = x^T A x \), \( A \) symmetric

\[ A = \begin{pmatrix} \alpha & \beta \\ \beta & 2 \end{pmatrix} \]

\[ x^T A x = (x_1, x_2) \begin{pmatrix} \alpha & \beta \\ \beta & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]
so $A$ is symmetric and

$$x^\top Ax = (x_1, x_2) \begin{pmatrix} \alpha & \frac{\beta}{2} \\ \frac{\beta}{2} & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= (x_1, x_2) \begin{pmatrix} \alpha x_1 + \frac{\beta x_2}{2} \\ \frac{\beta}{2} x_1 + \gamma x_2 \end{pmatrix}$$

$$= \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2$$

$$= f(x)$$

Notice $f(0) = 0$.

Can we determine anything about $f(x)$ for $x \neq 0$?

E.g., $f(x) > 0$ for $x$? Easy if $\beta = 0$...
Quadratic Forms

Consider a quadratic form

\[ f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \alpha_{ii} x_i^2 + \sum_{i<j} \beta_{ij} x_i x_j \]  

(1)

Let

\[ \alpha_{ij} = \begin{cases} \frac{\beta_{ij}}{2} & \text{if } i < j \\ \frac{\beta_{ii}}{2} & \text{if } i > j \end{cases} \]

Let

\[ A = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix} \]

so \( f(x) = x^\top A x \)

real symmetric
Quadratic Forms

$A$ is symmetric, so let $V = \{v_1, \ldots, v_n\}$ be an orthonormal basis of eigenvectors of $A$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$.

Then $A = U^\top D U = U^{-\top} D U$

where $D = \begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{pmatrix}$

and $U = Mtx_{V,W}(id)$ is unitary

The columns of $U^\top$ (the rows of $U$) are the coordinates of $v_1, \ldots, v_n$, expressed in terms of the standard basis $W$. Given $x \in \mathbb{R}^n$, recall

$$x = \sum_{i=1}^{n} \gamma_i v_i \text{ where } \gamma_i = x \cdot v_i$$
Quadratic Forms

So

\[ f(x) = f \left( \sum \gamma_i v_i \right) \]
\[ = (\sum \gamma_i v_i)^T A \left( \sum \gamma_i v_i \right) \]
\[ = (\sum \gamma_i v_i)^T U^T D U \left( \sum \gamma_i v_i \right) \]
\[ = (U \sum \gamma_i v_i)^T D \left( U \sum \gamma_i v_i \right) \]
\[ = \left( \sum \gamma_i U v_i \right)^T D \left( \sum \gamma_i U v_i \right) \]
\[ = (\gamma_1, \ldots, \gamma_n) D \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix} \]
\[ = \sum \lambda_i \gamma_i^2 \]

\[ \{ \text{eigenvalues of } A \} \]

\[ (EF)^T = F^T E^T \]
\[ (U \text{ is change of basis from } W \text{ to } V) \]
\[ \forall v \in U v_i = e_i = (0, \ldots, 1, 0, \ldots) \]
Quadratic Forms

The equation for a level set of $f$ is

$$\forall \gamma \in \mathbb{R}^n : f(\gamma) = C^2 = \left\{ \gamma \in \mathbb{R}^n : \sum_{i=1}^{n} \lambda_i \gamma_i^2 = C \right\}$$

- If $\lambda_i \geq 0$ for all $i$, the level set is an ellipsoid, with principal axes in the directions $v_1, \ldots, v_n$. The length of the principal axis along $v_i$ is $\sqrt{C/\lambda_i}$ if $C \geq 0$ (if $\lambda_i = 0$, the level set is a degenerate ellipsoid with principal axis of infinite length in that direction). The level set is empty if $C < 0$.

  \[ \Rightarrow \text{$f$ has global min at 0, } f(x) \geq 0 \ \forall x \]

- If $\lambda_i \leq 0$ for all $i$, the level set is an ellipsoid, with principal axes in the directions $v_1, \ldots, v_n$. The length of the principal

  \[ \Rightarrow \text{$f$ has global max at 0, } f(x) \leq 0 \ \forall x \]
axis along $v_i$ is $\sqrt{C/\lambda_i}$ if $C \leq 0$ (if $\lambda_i = 0$, the level set is a degenerate ellipsoid with principal axis of infinite length in that direction). The level set is empty if $C > 0$.

• If $\lambda_i > 0$ for some $i$ and $\lambda_j < 0$ for some $j$, the level set is a hyperboloid. For example, suppose $n = 2$, $\lambda_1 > 0$, $\lambda_2 < 0$. The equation is

$$C = \lambda_1 \gamma_1^2 + \lambda_2 \gamma_2^2$$

$$= \left(\sqrt{\lambda_1} \gamma_1 + \sqrt{|\lambda_2|} \gamma_2\right) \left(\sqrt{\lambda_1} \gamma_1 - \sqrt{|\lambda_2|} \gamma_2\right)$$

$\Rightarrow$ if has a saddle point at 0

$\min$ with respect to $v_i$

$max$ with respect to $v_j$
This is a hyperbola with asymptotes

\[
0 = \sqrt{\lambda_1 \gamma_1} + \sqrt{|\lambda_2| \gamma_2}
\]

\[\Rightarrow \gamma_1 = -\sqrt{\frac{|\lambda_2|}{\lambda_1}} \gamma_2\]

\[
0 = \left(\sqrt{\lambda_1 \gamma_1} - \sqrt{|\lambda_2| \gamma_2}\right)
\]

\[\Rightarrow \gamma_1 = \sqrt{\frac{|\lambda_2|}{\lambda_1}} \gamma_2\]
$\lambda_1 > 0, \lambda_2 > 0$

$f$ has a global min at $0$
\[ \gamma_1 > 0, \gamma_2 < 0 \]

\[ \gamma_1 = \sqrt{|\lambda_2|/\lambda_1} \]

\[ \gamma_2 = -\sqrt{|\lambda_2|/\lambda_1} \]

\[ \exists \gamma \in \mathbb{R}^n : f(x) = 0 \]

\( f \) was a saddle point at 0
Quadratic Forms

This proves the following corollary of Theorem 2.

**Corollary 1.** Consider the quadratic form (1). Let \( \{v_1, \ldots, v_n\} \) be an orthonormal basis of eigenvectors of \( A \) with corresponding eigenvalues \( \{\lambda_1, \ldots, \lambda_n\} \).

1. \( f \) has a global minimum at 0 if and only if \( \lambda_i \geq 0 \) for all \( i \); the level sets of \( f \) are ellipsoids with principal axes aligned with the orthonormal eigenvectors \( v_1, \ldots, v_n \).

2. \( f \) has a global maximum at 0 if and only if \( \lambda_i \leq 0 \) for all \( i \); the level sets of \( f \) are ellipsoids with principal axes aligned with the orthonormal eigenvectors \( v_1, \ldots, v_n \).
3. If $\lambda_i < 0$ for some $i$ and $\lambda_j > 0$ for some $j$, then $f$ has a saddle point at 0; the level sets of $f$ are hyperboloids with principal axes aligned with the orthonormal eigenvectors $v_1, \ldots, v_n$. 
Bounded Linear Maps

**Definition 3.** Suppose $X, Y$ are normed vector spaces and $T \in L(X, Y)$. We say $T$ is bounded if

$$\exists \beta \in \mathbb{R} \text{ s.t. } \|T(x)\|_Y \leq \beta \|x\|_X \quad \forall x \in X$$

Note this implies that $T$ is Lipschitz with constant $\beta$. 

Why not previous notion of bounded:

$$\exists \beta \in \mathbb{R} \text{ s.t. } \|T(x)\| \leq \beta \quad ?$$

$$T(\alpha x) = \alpha T(x) \quad \forall x \in \mathbb{R}$$

$$\Rightarrow \quad \|T(\alpha x)\| = |\alpha| \|T(x)\| \quad \forall x \in \mathbb{R}$$
Bounded Linear Maps

Much more is true:

**Theorem 3** (Thms. 4.1, 4.3). Let $X$ and $Y$ be normed vector spaces and $T \in L(X,Y)$. Then

- $T$ is continuous at some point $x_0 \in X$ if and only if $T$ is continuous at every $x \in X$.
- $T$ is uniformly continuous on $X$ if and only if $T$ is Lipschitz.
- $T$ is Lipschitz if and only if $T$ is bounded.

*Proof.* Suppose $T$ is continuous at $x_0$. Fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$
\|z - x_0\| < \delta \Rightarrow \|T(z) - T(x_0)\| < \varepsilon
$$
Now suppose $x$ is any element of $X$. If $\|y - x\| < \delta$, let $z = y - x + x_0$, so $\|z - x_0\| = \|y - x\| < \delta$.

\[
\|T(y) - T(x)\| = \|T(y - x)\| < \delta
\]

Then

\[
\|T(y - x + x_0 - x_0)\| = \|T(z - x_0)\|
\]

which proves that $T$ is continuous at every $x$, and uniformly continuous.

We claim that $T$ is bounded if and only if $T$ is continuous at $0$. Suppose $T$ is not bounded. Then

\[
\exists \{x_n\} \text{ s.t. } \|T(x_n)\| > n\|x_n\| \quad \forall n
\]
Note that $x_n \neq 0$. Let $\varepsilon = 1$. Fix $\delta > 0$ and choose $n$ such that $\frac{1}{n} < \delta$. Let

$$
\frac{x_n}{n\|x_n\|} = \frac{x_n}{n\|x_n\|} = \frac{1}{n} < \delta
$$

$$
\|x_n' - 0\| = \frac{1}{n\|x_n\|} \quad = \frac{1}{n\|x_n\|}
$$

$$
\|T(x_n') - T(0)\| = \frac{1}{n\|x_n\|} \|T(x_n)\| > \frac{n\|x_n\|}{n\|x_n\|} = 1 = \varepsilon
$$

$$
\left( \text{defn of } x_n' \right)
\left( + \text{ linear } \right)
\left( \text{defn of } x_n \right)
$$
Since this is true for every $\delta$, $T$ is not continuous at 0. Therefore, $T$ continuous at 0 implies $T$ is bounded. Now, suppose $T$ is bounded, so find $M$ such that $\|T(x)\| \leq M\|x\|$ for every $x \in X$. Given $\varepsilon > 0$, let $\delta = \varepsilon/M$. Then

$$\|x - 0\| < \delta \Rightarrow \|x\| < \delta$$

$$\Rightarrow \|T(x) - T(0)\| = \|T(x)\| < M\delta$$

$$\Rightarrow \|T(x) - T(0)\| < \varepsilon = M\delta$$

so $T$ is continuous at 0.

Thus, we have shown that continuity at some point $x_0$ implies uniform continuity, which implies continuity at every point, which implies $T$ is continuous at 0, which implies that $T$ is bounded, which implies that $T$ is continuous at 0, which implies that $T$ is
continuous at some $x_0$, so all of the statements except possibly the Lipschitz statement are equivalent.

Suppose $T$ is bounded, with constant $M$. Then

$$
\| T(x) - T(y) \| = \| T(x - y) \| \leq M \| x - y \|
$$

so $T$ is Lipschitz with constant $M$; conversely, if $T$ is Lipschitz with constant $M$, then $T$ is bounded with constant $M$. So all the statements are equivalent. \qed

\[ \forall x \in X: \ | T(x) - T(y) | = | T(x) | \leq M | x - y | \]
Bounded Linear Maps

Every linear map on a finite-dimensional normed vector space is bounded (and thus continuous, uniformly continuous, and Lipschitz continuous).

**Theorem 4** (Thm. 4.5). Let $X$ and $Y$ be normed vector spaces, with $\dim X = n$. Every $T \in L(X, Y)$ is bounded.

*Proof.* See de la Fuente. \qed
Topological Isomorphism

Definition 4. A topological isomorphism between normed vector spaces $X$ and $Y$ is a linear transformation $T \in L(X, Y)$ that is invertible (one-to-one, onto), continuous, and has a continuous inverse.

Two normed vector spaces $X$ and $Y$ are topologically isomorphic if there is a topological isomorphism $T : X \rightarrow Y$. 
The Space \( B(X, Y) \)

Suppose \( X \) and \( Y \) are normed vector spaces. We define

\[
B(X, Y) = \{ T \in L(X, Y) : T \text{ is bounded} \}
\]

\[
\|T\|_{B(X, Y)} = \sup \left\{ \frac{\|T(x)\|_Y}{\|x\|_X}, x \in X, x \neq 0 \right\}
\]

\[
= \sup \{\|T(x)\|_Y : \|x\|_X = 1 \}
\]

\[\Rightarrow \quad \|T(x)\| = \|T(\|x\|x)\| \leq \beta \|x\| \quad \forall x \neq 0 \]

We skip the proofs of the rest of these results – read dIF.
The Space $B(X,Y)$

**Theorem 5** (Thm. 4.8). Let $X, Y$ be normed vector spaces. Then

$$\left( B(X,Y), \| \cdot \|_{B(X,Y)} \right)$$

is a normed vector space.
The Space $B(\mathbb{R}^n, \mathbb{R}^m)$

**Theorem 6 (Thm. 4.9).** Let $T \in L(\mathbb{R}^n, \mathbb{R}^m)$ ($= B(\mathbb{R}^n, \mathbb{R}^m)$) with matrix $A = (a_{ij})$ with respect to the standard bases. Let

$$M = \max\{|a_{ij}| : 1 \leq i \leq m, 1 \leq j \leq n\}$$

Then

$$M \leq \|T\| \leq M\sqrt{mn}$$
Compositions

**Theorem 7** (Thm. 4.10). Let $R \in L(\mathbb{R}^m, \mathbb{R}^n)$ and $S \in L(\mathbb{R}^n, \mathbb{R}^p)$. Then

$$\|S \circ R\| \leq \|S\| \|R\|$$
Invertibility

Define $\Omega(\mathbb{R}^n) = \{ T \in L(\mathbb{R}^n, \mathbb{R}^n) : T \text{ is invertible} \}$

**Theorem 8** (Thm. 4.11'). Suppose $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $E$ is the standard basis of $\mathbb{R}^n$. Then

$T$ is invertible

$\iff \ker T = \{0\}$
$\iff \det (\text{Mat}_E(T)) \neq 0$
$\iff \det (\text{Mat}_V(T)) \neq 0$ for every basis $V$
$\iff \det (\text{Mat}_{V,W}(T)) \neq 0$ for every pair of bases $V, W$
Invertibility

Theorem 9 (Thm. 4.12). If $S, T \in \Omega(\mathbb{R}^n)$, then $S \circ T \in \Omega(\mathbb{R}^n)$ and

$$(S \circ T)^{-1} = T^{-1} \circ S^{-1}$$
Invertibility

**Theorem 10** (Thm. 4.14). Let \( S, T \in L(\mathbb{R}^n, \mathbb{R}^n) \). If \( T \) is invertible and
\[
\|T - S\| < \frac{1}{\|T^{-1}\|}
\]
then \( S \) is invertible. In particular, \( \Omega(\mathbb{R}^n) \) is open in \( L(\mathbb{R}^n, \mathbb{R}^n) = B(\mathbb{R}^n, \mathbb{R}^n) \).

**Theorem 11** (Thm. 4.15). The function \( (\cdot)^{-1} : \Omega(\mathbb{R}^n) \to \Omega(\mathbb{R}^n) \) that assigns \( T^{-1} \) to each \( T \in \Omega(\mathbb{R}^n) \) is continuous.