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Fixed Points for Functions

Definition 1. Let $X$ be a nonempty set and $f : X \to X$. A point $x^* \in X$ is a fixed point of $f$ if $f(x^*) = x^*$.

$x^*$ is a fixed point of $f$ if it is “fixed” by the map $f$.
Fixed Points for Functions

Examples:

1. Let $X = \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2x$. Then $x = 0$ is a fixed point of $f$ (and is the unique fixed point of $f$).

2. Let $X = \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x$. Then every point in $\mathbb{R}$ is a fixed point of $f$ (in particular, fixed points need not be unique).

3. Let $X = \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x + 1$. Then $f$ has no fixed points.
4. Let $X = [0, 2]$ and $f : X \to X$ be given by $f(x) = \frac{1}{2}(x + 1)$. Then

$$
\begin{align*}
    f(x) &= \frac{1}{2}(x + 1) = x \\
    \iff & \quad x + 1 = 2x \\
    \iff & \quad x = 1
\end{align*}
$$

So $x = 1$ is the unique fixed point of $f$. Notice that $f$ is a contraction (why?), so we already knew that $f$ must have a unique fixed point on $\mathbb{R}$ from the Contraction Mapping Theorem.

5. Let $X = [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$ and $f : X \to X$ be given by $f(x) = 1 - x$. Then $f$ has no fixed points.
6. Let $X = [-2, 2]$ and $f : X \to X$ be given by $f(x) = \frac{1}{2}x^2$. Then $f$ has two fixed points, $x = 0$ and $x = 2$. If instead $X' = (0, 2)$, then $f : X' \to X'$ but $f$ has no fixed points on $X'$.

7. Let $X = \{1, 2, 3\}$ and $f : X \to X$ be given by $f(1) = 2, f(2) = 3, f(3) = 1$ (so $f$ is a permutation of $X$). Then $f$ has no fixed points.

8. Let $X = [0, 2]$ and $f : X \to X$ be given by

$$f(x) = \begin{cases} 
  x + 1 & \text{if } x \leq 1 \\
  x - 1 & \text{if } x > 1
\end{cases}$$

Then $f$ has no fixed points.
A Simple Fixed Point Theorem

**Theorem 1.** Let $X = [a, b]$ for $a, b \in \mathbb{R}$ with $a < b$ and let $f : X \to X$ be continuous. Then $f$ has a fixed point.

**Proof.** Let $g : [a, b] \to \mathbb{R}$ be given by

$$g(x) = f(x) - x$$

If either $f(a) = a$ or $f(b) = b$, we’re done. So assume $f(a) > a$ and $f(b) < b$. Then

$$g(a) = f(a) - a > 0$$
$$g(b) = f(b) - b < 0$$

$g$ is continuous, so by the Intermediate Value Theorem, $\exists x^* \in (a, b)$ such that $g(x^*) = 0$, that is, such that $f(x^*) = x^*$. \qed
$g(x) = f(x) - x$
Brouwer’s Fixed Point Theorem

**Theorem 2** (Thm. 3.2. Brouwer’s Fixed Point Theorem). *Let* \( X \subseteq \mathbb{R}^n \) *be nonempty, compact, and convex, and let* \( f : X \rightarrow X \) *be continuous. Then* \( f \) *has a fixed point.*
Sketch of Proof of Brouwer

Consider the case when the set $X$ is the unit ball in $\mathbb{R}^n$, i.e. $X = B_1[0] = B = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. Let $f : B \to B$ be a continuous function. Recall that $\partial B$ denotes the boundary of $B$, so $\partial B = \{x \in \mathbb{R}^n : \|x\| = 1\}$.

**Fact:** Let $B$ be the unit ball in $\mathbb{R}^n$. Then there is no continuous function $h : B \to \partial B$ such that $h(x') = x'$ for every $x' \in \partial B$.

See J. Franklin, Methods of Mathematical Economics, for an elementary (but long) proof.
Now to establish Brouwer’s theorem, suppose, by way of contradiction, that $f$ has no fixed points in $B$. Thus for every $x \in B$, $x \neq f(x)$.

Since $x \neq f(x)$ for every $x$, we can carry out the following construction. For each $x \in B$, construct the line segment originating at $f(x)$ and going through $x$. Let $g(x)$ denote the intersection of this line segment with $\partial B$.

This construction is well-defined, and gives a continuous function $g : B \to \partial B$. Furthermore, if $x' \in \partial B$, then $x' = g(x')$. That is, $g|_{\partial B} = \text{id}_{\partial B}$. Since there are no such functions by the fact above, we have a contradiction. Therefore there exists $x^* \in B$ such that $f(x^*) = x^*$, that is, $f$ has a fixed point in $B$. 
- $f(x) = x$
Fixed Points for Correspondences

**Definition 2.** Let $X$ be nonempty and $\Psi : X \rightarrow 2^X$ be a correspondence. A point $x^* \in X$ is a fixed point of $\Psi$ if $x^* \in \Psi(x^*)$.

Note here that we do not require $\Psi(x^*) = \{x^*\}$, that is $\Psi$ need not be single-valued at $x^*$. So $x^*$ can be a fixed point of $\Psi$ but there may be other elements of $\Psi(x^*)$ different from $x^*$. 
Examples:

1. Let $X = [0, 4]$ and $\Psi : X \to 2^X$ be given by

$$\Psi(x) = \begin{cases} 
[x + 1, x + 2] & \text{if } x < 2 \\
[0, 4] & \text{if } x = 2 \\
[x - 2, x - 1] & \text{if } x > 2
\end{cases}$$

Then $x = 2$ is the unique fixed point of $\Psi$.

2. Let $X = [0, 4]$ and $\Psi : X \to 2^X$ be given by

$$\Psi(x) = \begin{cases} 
[x + 1, x + 2] & \text{if } x < 2 \\
[0, 1] \cup [3, 4] & \text{if } x = 2 \\
[x - 2, x - 1] & \text{if } x > 2
\end{cases}$$

Then $\Psi$ has no fixed points.
graph \( \Psi \)
Kakutani’s Fixed Point Theorem

Theorem 3. (Thm. 3.4’. Kakutani’s Fixed Point Theorem)
Let $X \subseteq \mathbb{R}^n$ be a non-empty, compact, convex set and $\Psi : X \to 2^X$ be an upper hemi-continuous correspondence with non-empty, convex, compact values. Then $\Psi$ has a fixed point in $X$.

Proof. (sketch) Here, the idea is to use Brouwer’s theorem after appropriately approximating the correspondence with a function. The catch is that there won’t necessarily exist a continuous selection from $\Psi$, that is, a continuous function $f : X \to X$ such that $f(x) \in \Psi(x)$ for every $x \in X$. If such a function existed, then by applying Brouwer to $f$ we would have a fixed point of $\Psi$ (because if $\exists x^* \in X$ such that $x^* = f(x^*)$, then $x^* = f(x^*) \in \Psi(x^*)$).
A graph labeled $\text{graph } \Psi$ with the line $y = x$ and shaded regions.
Instead, we look for a weaker type of approximation. Let \( X \subset \mathbb{R}^n \) be a non-empty, compact, convex set, and let \( \Psi : X \rightarrow 2^X \) be an uhc correspondence with non-empty, compact, convex values. For every \( \varepsilon > 0 \), define the \( \varepsilon \) ball about \( \text{graph } \Psi \) to be

\[
B_{\varepsilon}(\text{graph } \Psi) = \left\{ z \in X \times X : d(z, \text{graph } \Psi) = \inf_{(x,y) \in \text{graph } \Psi} d(z, (x, y)) < \varepsilon \right\}
\]

Here \( d \) denotes the ordinary Euclidean distance. Since \( \Psi \) is a convex-valued correspondence, for every \( \varepsilon > 0 \) there exists a continuous function \( f_\varepsilon : X \rightarrow X \) such that \( \text{graph } f_\varepsilon \subseteq B_{\varepsilon}(\text{graph } \Psi) \).
Now by letting $\varepsilon \to 0$, this means that we can find a sequence of continuous functions $\{f_n\}$ such that $\operatorname{graph} f_n \subseteq B_{\frac{1}{n}}(\operatorname{graph} \Psi)$ for each $n$. By Brouwer’s Fixed Point Theorem, each function $f_n$ has a fixed point $\hat{x}_n \in X$, and

$$(\hat{x}_n, \hat{x}_n) = (\hat{x}_n, f_n(\hat{x}_n)) \in \operatorname{graph} f_n \subseteq B_{\frac{1}{n}}(\operatorname{graph} \Psi) \text{ for each } n$$

So for each $n$ there exists $(x_n, y_n) \in \operatorname{graph} \Psi$ such that

$$d(\hat{x}_n, x_n) < \frac{1}{n} \text{ and } d(\hat{x}_n, y_n) < \frac{1}{n}$$

Since $X$ is compact, $\{\hat{x}_n\}$ has a convergent subsequence $\{\hat{x}_{n_k}\}$, with $\hat{x}_{n_k} \to \hat{x} \in X$. Then $x_{n_k} \to \hat{x}$ and $y_{n_k} \to \hat{x}$. Since $\Psi$ is uhc and closed-valued, it has closed graph, so $(\hat{x}, \hat{x}) \in \operatorname{graph} \Psi$. Thus $\hat{x} \in \Psi(\hat{x})$, that is, $\hat{x}$ is a fixed point of $\Psi$. \qed
Separating Hyperplane Theorems

**Theorem 4** (1.26, Separating Hyperplane Theorem). Let $A, B \subseteq \mathbb{R}^n$ be nonempty, disjoint convex sets. Then there exists a nonzero vector $p \in \mathbb{R}^n$ such that

$$p \cdot a \leq p \cdot b \quad \forall a \in A, b \in B$$
Separating a Point from a Set

**Theorem 5.** Let $Y \subseteq \mathbb{R}^n$ be a nonempty convex set and $x \notin Y$. Then there exists a nonzero vector $p \in \mathbb{R}^n$ such that

$$p \cdot x \leq p \cdot y \quad \forall y \in Y$$

**Proof.** We sketch the proof in the special case that $Y$ is compact. We will see that in this case we actually get a stronger conclusion:

$$\exists p \in \mathbb{R}^n, p \neq 0 \text{ s.t. } p \cdot x < p \cdot y \quad \forall y \in Y$$

Choose $y_0 \in Y$ such that $|y_0 - x| = \inf\{|y - x| : y \in Y\}$; such a point exists because $Y$ is compact, so the distance function $g(y) = |y - x|$ assumes its minimum on $Y$. Since $x \notin Y$, $x \neq y_0$, so $y_0 - x \neq 0$. Let $p = y_0 - x$. The set

$$H = \{z \in \mathbb{R}^n : p \cdot z = p \cdot y_0\}$$
is the hyperplane perpendicular to $p$ through $y_0$. See Figure 12. Then

$$
p \cdot y_0 = (y_0 - x) \cdot y_0
$$
$$
= (y_0 - x) \cdot (y_0 - x + x)
$$
$$
= (y_0 - x) \cdot (y_0 - x) + (y_0 - x) \cdot x
$$
$$
= |y_0 - x|^2 + p \cdot x
$$
$$
> p \cdot x
$$

We claim that

$$
y \in Y \Rightarrow p \cdot y \geq p \cdot y_0
$$

If not, suppose there exists $y \in Y$ such that $p \cdot y < p \cdot y_0$. Given $\alpha \in (0, 1)$, let

$$
w_\alpha = \alpha y + (1 - \alpha)y_0$$
Since $Y$ is convex, $w_{\alpha} \in Y$. Then for $\alpha$ sufficiently close to zero,

$$|x - w_{\alpha}|^2 = |x - \alpha y - (1 - \alpha)y_0|^2$$

$$= |x - y_0 + \alpha(y_0 - y)|^2$$

$$= |-p + \alpha(y_0 - y)|^2$$

$$= |p|^2 - 2\alpha p \cdot (y_0 - y) + \alpha^2 |y_0 - y|^2$$

$$= |p|^2 + \alpha \left( -2p \cdot (y_0 - y) + \alpha |y_0 - y|^2 \right)$$

$$< |p|^2 \quad \text{for } \alpha \text{ close to } 0, \text{ as } p \cdot y_0 > p \cdot y$$

$$= |y_0 - x|^2$$

Thus for $\alpha$ sufficiently close to zero,

$$|w_{\alpha} - x| < |y_0 - x|$$

which implies $y_0$ is not the closest point in $Y$ to $x$, contradiction. $\square$
$H = \{ z : p \cdot z = p \cdot y_0 \}$
The general version of the Separating Hyperplane Theorem can be derived from this special case by noting that if $A \cap B = \emptyset$, then $0 \notin A - B = \{a - b : a \in A, b \in B\}$. 
Strict Separation

For the special case of $Y$ compact and $X = \{x\}$, we actually could strictly separate $Y$ and $X$:

$$\exists p \in \mathbb{R}^n, p \neq 0 \text{ s.t. } p \cdot x < p \cdot y \quad \forall y \in Y$$

When can we do this in general? Will require additional assumptions...
Strict Separation

Theorem 6. (Strict Separating Hyperplane Theorem) Let $A, B \subseteq \mathbb{R}^n$ be nonempty, disjoint, closed, convex sets. Then there exists a nonzero vector $p \in \mathbb{R}^n$ such that

$$p \cdot a < p \cdot b \quad \forall a \in A, b \in B$$