Econ 204 2017

Lecture 1

Outline

1. Introductions
2. About the Course and Other Administrative Details
3. Methods of Proof
4. Equivalence Relations
5. Cardinality

Announcements:
- Lect. 11F 11 - 1:30
- PS 1 posted later today
  ➔ due Friday
- no sections today
Introductions

Welcome

- 204

- Berkeley Economics

- UC Berkeley

- Berkeley

- California
• US...
Introductions

- Chris Shannon
- Farzad Pourbabaee
- Walker Ray
About the Course

• **Schedule:** Lectures MTWThF 9:00 - 11:30 141 McCone Hall, often going over so don’t schedule anything before 12:00

  **Sections:** MTWThF 1:00 - 2:30 and 2:30 - 4:00, in 648 Evans (please try to split up evenly)

  **Office hours:** Chris Shannon MTWThF 11:30 - 12:30 (end of lecture + 1 hour) here or 511 Evans, also by appt.

  Farzad + Walker MTWThF 4:00 - 5:00 648 Evans

• **Final Exam:** Wednesday August 9, 9:00 am - 12:00 pm, location TBA
- **Prerequisites:** Math 53-54 at Berkeley or equivalent
  - 4 semesters college mathematics
  - linear algebra
  - multivariable calculus
  - rigorous approach - theorems stated carefully and some proofs given
  - stream for engineers and scientists
Course requirements:

• problems sets: 6 total
  (no late problem sets...no exceptions)

• exam

• reading/working on your own

Grade: 10% problem sets (5 highest scores out of 6), 90% final exam
Grading in First Year Economics Courses:

- median grade = B+: solid command of material

- A and A- are very good grades, A+ for truly exceptional work

- B: ready to go on to further work...a B in 204 means you are ready to go on to 201a/b, 202a/b, 240a/b

- B- : very marginal, but we won’t make you take the class again. B- in 204 means you will have a very hard time in 201a/b. Recommend you take Math 53 and 54 this year, maybe Math 104, come back next year to retake 204 and
take 201a/b. B- is a passing grade, but you must maintain a B average

- **C**: not passing. Definitely not ready for 201a/b, 202a/b, 240a/b. Take Math 53-54 this year, maybe Math 104, retake 204 next year

- 204 with at least a B- (or a waiver from 204 requirement) is a strictly enforced prerequisite for enrollment in 201a/b

- **F**: means you didn’t take the final exam. Be sure to withdraw if you don’t or can’t take the final.
Resources:

Book: de la Fuente, *Mathematical Methods and Models for Economists*

Lecture notes: for every lecture + supplements for several topics

*Be sure to read Corrections Handout with dlF*

Seek out other references
This class is not normal...

- lectures
- expectations
- classroom stuff
Goals for 204

- reduce heterogeneity of math backgrounds for students in Econ graduate classes

- advance everyone’s math skills and knowledge

- present some particular concepts and results used in first-year economics courses 201a/b, 202a/b, 240a/b

- challenge everyone - so not everyone will understand everything
• develop basic math skills and knowledge needed to work as a professional economist and read academic economics

• develop ability to read and evaluate purported proofs...essential for reading and working in all branches of economics - theoretical, empirical, experimental

• develop ability to compose simple proofs...essential to working in all branches of economics - theoretical, empirical, experimental

• cover selected material from real analysis and linear algebra at moderate level of abstraction (considerably more advanced and abstract than Math 53 + 54)
• **not** to review Math 53 + 54. If you are weak on this material, take Math 53-54 this year, and take 204 next year.
Learning by Doing

- to learn this sort of mathematics you need to do more than just read the book and notes and listen to lectures

- active reading: work through each line, be sure you know how to get from one line to the next

- active listening: follow each step as we work through arguments in class

- working problems: the most valuable part of the class
• working in groups strongly encouraged...

• but, always try to work through all of the problems before talking to others

• everyone must write up his/her own solutions

• best test of understanding: can you explain it to others
Methods of Proof

What is a proof? The million dollar question...

Main Methods of Proof:

- deduction
- contraposition
- induction
- contradiction
We’ll examine each of these in turn.
Proof by Deduction

**Proof by Deduction:** A list of statements, the last of which is the statement to be proven. Each statement in the list is either

- an axiom: a fundamental assumption about mathematics, or part of definition of the object under study; or

- a previously established theorem; or

- follows from previous statements in the list by a valid rule of inference
Example: Prove that the function $f(x) = x^2$ is continuous at $x = 5$.

Recall from one-variable calculus that $f(x) = x^2$ is continuous at $x = 5$ means

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } |x - 5| < \delta \Rightarrow |f(x) - f(5)| < \varepsilon$$

That is, “for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $x$ is within $\delta$ of 5, $f(x)$ is within $\varepsilon$ of $f(5)$.”

To prove the claim, we must systematically verify that this definition is satisfied.
Proof. Let $\varepsilon > 0$ be given. Let

$$\delta = \min \left\{ 1, \frac{\varepsilon}{11} \right\} > 0$$

$\Rightarrow \quad \delta \geq \frac{\varepsilon}{11}$

*Where did that come from?* Suppose $|x - 5| < \delta$. Since $\delta \leq 1$, $4 < x < 6$, so $9 < x + 5 < 11$ and $|x + 5| < 11$. Then

$$|f(x) - f(5)| = |x^2 - 25|$$
$$= |(x + 5)(x - 5)|$$
$$= |x + 5||x - 5|$$
$$\leq 11 \cdot \delta$$
$$\leq 11 \cdot \frac{\varepsilon}{11}$$
$$= \varepsilon$$

Thus, we have shown that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - 5| < \delta \Rightarrow |f(x) - f(5)| < \varepsilon$, so $f$ is continuous at $x = 5$. $\square$
Proof by Contraposition

Recall some basics of logic.

\( \neg P \) means “\( P \) is false.”

\( P \land Q \) means “\( P \) is true and \( Q \) is true.”

\( P \lor Q \) means “\( P \) is true or \( Q \) is true (or possibly both).”

\( \neg P \land Q \) means \((\neg P) \land Q\); \( \neg P \lor Q \) means \((\neg P) \lor Q\).

\( P \Rightarrow Q \) means “whenever \( P \) is satisfied, \( Q \) is also satisfied.”

Formally, \( P \Rightarrow Q \) is equivalent to \( \neg P \lor Q \).
Proof by Contraposition

The *contrapositive* of the statement $P \Rightarrow Q$ is the statement $\neg Q \Rightarrow \neg P$.

**Theorem 1.** $P \Rightarrow Q$ is true if and only if $\neg Q \Rightarrow \neg P$ is true.

*Proof.* Suppose $P \Rightarrow Q$ is true. Then either $P$ is false, or $Q$ is true (or possibly both). Therefore, either $\neg P$ is true, or $\neg Q$ is false (or possibly both), so $(\neg Q) \lor (\neg P)$ is true, that is, $\neg Q \Rightarrow \neg P$ is true.

Conversely, suppose $\neg Q \Rightarrow \neg P$ is true. Then either $\neg Q$ is false, or $\neg P$ is true (or possibly both), so either $Q$ is true, or $P$ is false (or possibly both), so $\neg P \lor Q$ is true, so $P \Rightarrow Q$ is true. $\Box$
Proof by Induction

We illustrate with an example:

**Theorem 2.** For every \( n \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\} \),

\[
\sum_{k=1}^{n} k = \frac{n(n + 1)}{2}
\]

i.e. \( 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \).

**Proof.** **Base step** \( n = 0 \): LHS = \( \sum_{k=1}^{0} k = \) the empty sum = 0. RHS = \( \frac{0 \cdot 1}{2} = 0 \)

So the claim is true for \( n = 0 \).
**Induction step:** Suppose

\[ \sum_{k=1}^{n} k = \frac{n(n + 1)}{2} \text{ for some } n \geq 0 \]

We must show that

\[ \sum_{k=1}^{n+1} k = \frac{(n + 1)((n + 1) + 1)}{2} \]
\[ \text{LHS} = \sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n + 1) \]
\[ = \frac{n(n + 1)}{2} + (n + 1) \text{ by the Induction hypothesis} \]
\[ = (n + 1) \left( \frac{n}{2} + 1 \right) \]
\[ = (n + 1)(n + 2) \]
\[ \text{RHS} = \frac{(n + 1)((n + 1) + 1)}{2} \]
\[ = \frac{(n + 1)(n + 2)}{2} = \text{LHS} \]

So by mathematical induction, \( \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \) for all \( n \in \mathbb{N}_0 \). \( \square \)
Proof by Contradiction

Assume the negation of what is claimed, and work toward a contradiction.

**Theorem 3.** There is no rational number \( q \) such that \( q^2 = 2 \).

**Proof.** Suppose \( q^2 = 2 \) where \( q \in \mathbb{Q} \). Then we can write \( q = \frac{m}{n} \) for some integers \( m, n \in \mathbb{Z} \). Moreover, we can assume that \( m \) and \( n \) have no common factor; if they did, we could divide it out.

\[
2 = q^2 = \frac{m^2}{n^2}
\]

Therefore, \( m^2 = 2n^2 \), so \( m^2 \) is even.
We claim that \( m \) is even. If not, then \( m \) is odd, so \( m = 2p + 1 \) for some \( p \in \mathbb{Z} \). Then
\[
m^2 = (2p + 1)^2
\]
\[
= 4p^2 + 4p + 1
\]
\[
= 2(2p^2 + 2p) + 1
\]
which is odd, contradiction. Therefore, \( m \) is even, so \( m = 2r \) for some \( r \in \mathbb{Z} \).
\[
4r^2 = (2r)^2
\]
\[
= m^2
\]
\[
= 2n^2
\]
\[
= 2r^2
\]
So \( n^2 \) is even, which implies (by the argument given above) that \( n \) is even. Therefore, \( n = 2s \) for some \( s \in \mathbb{Z} \), so \( m \) and \( n \) have a
common factor, namely 2, contradiction. Therefore, there is no rational number $q$ such that $q^2 = 2$. \qed
Equivalence Relations

**Definition 1.** A binary relation $R$ from $X$ to $Y$ is a subset $R \subseteq X \times Y$. We write $xRy$ if $(x, y) \in R$ and “not $xRy$” if $(x, y) \not\in R$. $R \subseteq X \times X$ is a binary relation on $X$.

**Example:** Suppose $f : X \to Y$ is a function from $X$ to $Y$. The binary relation $R \subseteq X \times Y$ defined by

$$xRy \iff f(x) = y$$

is exactly the graph of the function $f$. A function can be considered a binary relation $R$ from $X$ to $Y$ such that for each $x \in X$ there exists exactly one $y \in Y$ such that $(x, y) \in R$.

**Example:** Suppose $X = \{1, 2, 3\}$ and $R$ is the binary relation on $X$ given by $R = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3)\}$. This is the binary relation “is weakly greater than,” or $\geq$. 

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Equivalence Relations

**Definition 2.** A binary relation $R$ on $X$ is

(i) reflexive if $\forall x \in X, xRx$

(ii) symmetric if $\forall x, y \in X, xRy \iff yRx$

(iii) transitive if $\forall x, y, z \in X, (xRy \land yRz) \implies xRz$

**Definition 3.** A binary relation $R$ on $X$ is an equivalence relation if it is reflexive, symmetric and transitive.
\[ [x] = A \subseteq X \]
\[ \Sigma y = B \subseteq X \]

? y \in [x] = A

but \[ [y] = B \neq A \]

y \in [x] \iff A = B
Equivalence Relations

**Definition 4.** Given an equivalence relation $R$ on $X$, write

$$[x] = \{y \in X : xRy\}$$

$[x]$ is called the equivalence class containing $x$.

*The set of equivalence classes is the quotient of $X$ with respect to $R$, denoted $X/R$. "$X \mod R".*

**Example:** The binary relation $\geq$ on $\mathbb{R}$ is not an equivalence relation because it is not symmetric.

**Example:** Let $X = \{a, b, c, d\}$ and

$$R = \{(a, a), (a, b), (b, a), (b, b), (c, c), (c, d), (d, c), (d, d)\}$$

$R$ is an equivalence relation (why?) and the equivalence classes of $R$ are $\{a, b\}$ and $\{c, d\}$. $X/R = \{\{a, b\}, \{c, d\}\}$

$[a] = \{a, b\}$  \hspace{1cm}  $[c] = \{c, d\}$

$[b] = \{a, b\}$  \hspace{1cm}  $[d] = \{c, d\}$
Equivalence Relations

The equivalence classes of an equivalence relation form a partition of $X$: every element of $X$ belongs to exactly one equivalence class.

**Theorem 4.** Let $R$ be an equivalence relation on $X$. Then $\forall x \in X, x \in [x]$. Given $x, y \in X$, either $[x] = [y]$ or $[x] \cap [y] = \emptyset$.

**Proof.** If $x \in X$, then $xRx$ because $R$ is reflexive, so $x \in [x]$.

Suppose $x, y \in X$. If $[x] \cap [y] = \emptyset$, we’re done. So suppose $[x] \cap [y] \neq \emptyset$. We must show that $[x] = [y]$, i.e. that the elements of $[x]$ are exactly the same as the elements of $[y]$. 

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Choose \( z \in [x] \cap [y] \). Then \( z \in [x] \), so \( xRz \). By symmetry, \( zRx \).
Also \( z \in [y] \), so \( yRz \). By symmetry again, \( zRy \). Now choose \( w \in [x] \). By definition, \( xRw \). Since \( zRx \) and \( R \) is transitive, \( zRw \).
By symmetry, \( wRz \). Since \( zRy \), \( wRy \) by transitivity again. By symmetry, \( yRw \), so \( w \in [y] \), which shows that \( [x] \subseteq [y] \).
Similarly, \( [y] \subseteq [x] \), so \( [x] = [y] \).
Cardinality

**Definition 5.** Two sets $A, B$ are numerically equivalent (or have the same cardinality) if there is a bijection $f : A \rightarrow B$, that is, a function $f : A \rightarrow B$ that is 1-1 ($a \neq a' \Rightarrow f(a) \neq f(a')$), and onto ($\forall b \in B \ \exists a \in A \ s.t. \ f(a) = b$).

**Example:** $A = \{2, 4, 6, \ldots, 50\}$ is numerically equivalent to the set $\{1, 2, \ldots, 25\}$ under the function $f(n) = 2n$.

$B = \{1, 4, 9, 16, 25, 36, 49 \ldots\} = \{n^2 : n \in \mathbb{N}\}$ is numerically equivalent to $\mathbb{N}$. 
Cardinality

A set is either finite or infinite. A set is *finite* if it is numerically equivalent to \( \{1, \ldots, n\} \) for some \( n \). A set that is not finite is *infinite*.

In particular, \( A = \{2, 4, 6, \ldots, 50\} \) is finite, \( B = \{1, 4, 9, 16, 25, 36, 49 \ldots\} \) is infinite.

A set is *countable* if it is numerically equivalent to the set of natural numbers \( \mathbb{N} = \{1, 2, 3, \ldots\} \). An infinite set that is not countable is called *uncountable*.
Cardinality

Example: The set of integers \( \mathbb{Z} \) is countable.

\[
\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}
\]

Define \( f : \mathbb{N} \rightarrow \mathbb{Z} \) by

\[
\begin{align*}
f(1) &= 0 \\
f(2) &= 1 \\
f(3) &= -1 \\
&\vdots \\
f(n) &= (-1)^n \left\lfloor \frac{n}{2} \right\rfloor
\end{align*}
\]

where \( \lfloor x \rfloor \) is the greatest integer less than or equal to \( x \). It is straightforward to verify that \( f \) is one-to-one and onto.
Cardinality

Theorem 5. The set of rational numbers $\mathbb{Q}$ is countable.

“Picture Proof”:

$$Q = \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\}$$

$$= \left\{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N} \right\}$$
Go back and forth on upward-sloping diagonals, omitting the
repeats:

\begin{align*}
  f(1) &= 0 \\
  f(2) &= 1 \\
  f(3) &= \frac{1}{2} \\
  f(4) &= -1 \\
  &\vdots
\end{align*}

\( f: \mathbb{N} \to \mathbb{Q} \), \( f \) is one-to-one and onto.