

# Econ 204 2017

## Lecture 7

### Outline

1. Connected Sets
2. Correspondences
3. Continuity for Correspondences

# Connected Sets

**Definition 1.** *Two sets  $A, B$  in a metric space are separated if*

$$\bar{A} \cap B = A \cap \bar{B} = \emptyset$$

*A set in a metric space is connected if it cannot be written as the union of two nonempty separated sets.*

# Connected Sets

**Example:**  $[0, 1)$  and  $[1, 2]$  are disjoint but not separated:

$$\overline{[0, 1)} \cap [1, 2] = [0, 1] \cap [1, 2] = \{1\} \neq \emptyset$$

$[0, 1)$  and  $(1, 2]$  are separated:

$$\overline{[0, 1)} \cap (1, 2] = [0, 1] \cap (1, 2] = \emptyset$$

$$[0, 1) \cap \overline{(1, 2]} = [0, 1) \cap [1, 2] = \emptyset$$

Note that  $d([0, 1), (1, 2]) = 0$  even though the sets are separated.

# Connected Sets

- Note that separation does *not* require that  $\bar{A} \cap \bar{B} = \emptyset$ .  
For example,

$$[0, 1) \cup (1, 2]$$

is not connected.

- A common equivalent definition: A set  $Y$  in a metric space  $X$  is connected if there do not exist open sets  $A$  and  $B$  such that  $A \cap B = \emptyset$ ,  $Y \subseteq A \cup B$  and  $A \cap Y \neq \emptyset$  and  $B \cap Y \neq \emptyset$ .

## Connected Sets

**Theorem 1** (Thm. 9.2). *A set  $S \subseteq \mathbf{E}^1$  is connected if and only if it is an interval, i.e. if  $x, y \in S$  and  $z \in (x, y)$ , then  $z \in S$ .*

*Proof.* First, we show that if  $S$  is connected then  $S$  is an interval. We do this by proving the contrapositive: if  $S$  is not an interval, then it is not connected. If  $S$  is not an interval, find

$$x, y \in S, \quad x < z < y, \quad z \notin S$$

Let

$$A = S \cap (-\infty, z), \quad B = S \cap (z, \infty)$$

Then

$$\bar{A} \cap B \subseteq \overline{(-\infty, z)} \cap (z, \infty) = (-\infty, z] \cap (z, \infty) = \emptyset$$

$$A \cap \bar{B} \subseteq (-\infty, z) \cap \overline{(z, \infty)} = (-\infty, z) \cap [z, \infty) = \emptyset$$

$$A \cup B = (S \cap (-\infty, z)) \cup (S \cap (z, \infty))$$

$$= S \setminus \{z\}$$

$$= S$$

$$x \in A, \text{ so } A \neq \emptyset$$

$$y \in B, \text{ so } B \neq \emptyset$$

So  $S$  is not connected. We have shown that if  $S$  is not an interval, then  $S$  is not connected; therefore, if  $S$  is connected, then  $S$  is an interval.

Now, we need to show that if  $S$  is an interval, it is connected. This is much like the proof of the Intermediate Value Theorem. See de la Fuente for the details.  $\square$

# Continuity and Connectedness

In a general metric space, continuity will preserve connectedness.

**Theorem 2** (Thm. 9.3). *Let  $X$  be a metric space and  $f : X \rightarrow Y$  be continuous. If  $C$  is a connected subset of  $X$ , then  $f(C)$  is connected.*

*Proof.* We prove the contrapositive: if  $f(C)$  is not connected, then  $C$  is not connected. Suppose  $f(C)$  is not connected. Then there exist  $P, Q$  such that  $P \neq \emptyset \neq Q$ ,  $f(C) = P \cup Q$ , and

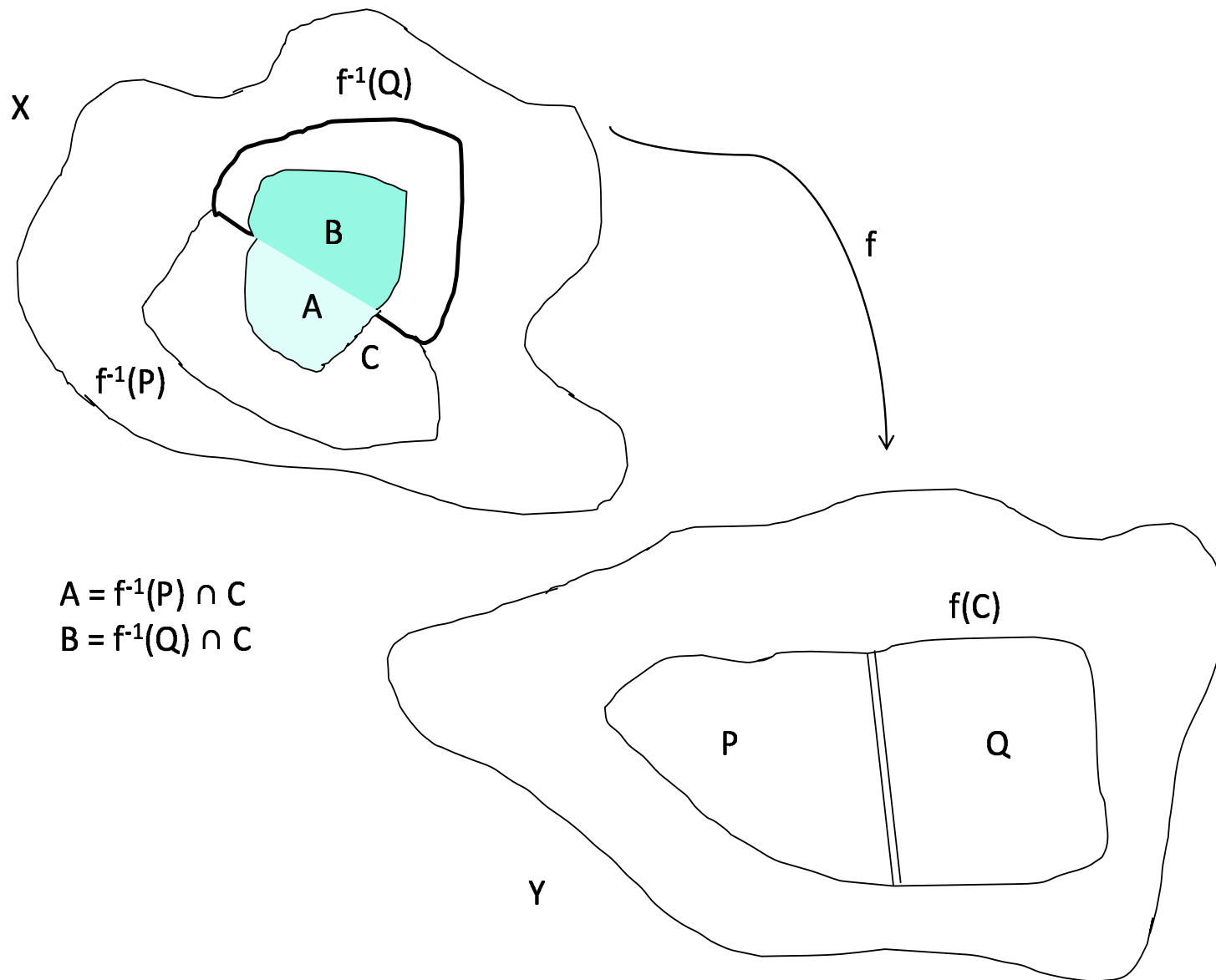
$$\bar{P} \cap Q = P \cap \bar{Q} = \emptyset$$

Let

$$A = f^{-1}(P) \cap C \text{ and } B = f^{-1}(Q) \cap C$$







$$A = f^{-1}(P) \cap C$$
$$B = f^{-1}(Q) \cap C$$

Then

$$\begin{aligned} A \cup B &= (f^{-1}(P) \cap C) \cup (f^{-1}(Q) \cap C) \\ &= (f^{-1}(P) \cup f^{-1}(Q)) \cap C \\ &= f^{-1}(P \cup Q) \cap C \\ &= f^{-1}(f(C)) \cap C \\ &= C \end{aligned}$$

Also,  $A = f^{-1}(P) \cap C \neq \emptyset$  and  $B = f^{-1}(Q) \cap C \neq \emptyset$ . Then note

$$A = f^{-1}(P) \cap C \subseteq f^{-1}(P) \subseteq f^{-1}(\bar{P})$$

Since  $f$  is continuous,  $f^{-1}(\bar{P})$  is closed, so

$$\bar{A} \subseteq f^{-1}(\bar{P})$$

Similarly,

$$B = f^{-1}(Q) \cap C \subseteq f^{-1}(Q) \subseteq f^{-1}(\bar{Q})$$

and  $f^{-1}(\bar{Q})$  is closed, so

$$\bar{B} \subseteq f^{-1}(\bar{Q})$$

Then

$$\begin{aligned} \bar{A} \cap B &\subseteq f^{-1}(\bar{P}) \cap f^{-1}(Q) \\ &= f^{-1}(\bar{P} \cap Q) \\ &= f^{-1}(\emptyset) \\ &= \emptyset \end{aligned}$$

and similarly

$$\begin{aligned} A \cap \bar{B} &\subseteq f^{-1}(P) \cap f^{-1}(\bar{Q}) \\ &= f^{-1}(P \cap \bar{Q}) \\ &= f^{-1}(\emptyset) \\ &= \emptyset \end{aligned}$$

So  $C$  is not connected. We have shown that  $f(C)$  not connected implies  $C$  not connected; therefore,  $C$  connected implies  $f(C)$  connected.  $\square$

You can view this result as a generalization of the Intermediate Value Theorem.

# Intermediate Value Theorem, Yet Again

This lets us give a third, and slickest, proof of the Intermediate Value Theorem.

**Corollary 1** (Intermediate Value Theorem). *If  $f : [a, b] \rightarrow \mathbf{R}$  is continuous, and  $f(a) < d < f(b)$ , then there exists  $c \in (a, b)$  such that  $f(c) = d$ .*

*Proof.* Since  $[a, b]$  is an interval, it is connected. So  $f([a, b])$  is connected, hence  $f([a, b])$  is an interval.  $f(a) \in f([a, b])$ , and  $f(b) \in f([a, b])$ , and  $d \in [f(a), f(b)]$ ; since  $f([a, b])$  is an interval,  $d \in f([a, b])$ , i.e. there exists  $c \in [a, b]$  such that  $f(c) = d$ . Since  $f(a) < d < f(b)$ ,  $c \neq a$ ,  $c \neq b$ , so  $c \in (a, b)$ .  $\square$

# Correspondences

**Definition 2.** A correspondence  $\psi : X \rightarrow 2^Y$  from  $X$  to  $Y$  is a function from  $X$  to  $2^Y$ , that is,  $\psi(x) \subseteq Y$  for every  $x \in X$ .

# Correspondences

## Examples:

1. Let  $u : \mathbf{R}_+^n \rightarrow \mathbf{R}$  be a continuous utility function,  $y > 0$  and  $p \in \mathbf{R}_{++}^n$ , that is,  $p_i > 0$  for each  $i$ .

Define  $\Psi : \mathbf{R}_{++}^n \times \mathbf{R}_{++} \rightarrow 2^{\mathbf{R}_+^n}$  by

$$\begin{aligned}\Psi(p, y) = \arg \max u(x) \\ \text{s.t. } x \geq 0 \\ p \cdot x \leq y\end{aligned}$$

$\Psi$  is the *demand correspondence* associated with the utility function  $u$ ; typically  $\Psi(p, y)$  is multi-valued.

2. Let  $f : X \rightarrow Y$  be a function. Define  $\Psi : X \rightarrow 2^Y$  by

$$\Psi(x) = \{f(x)\} \text{ for each } x \in X$$

That is, we can consider a function to be the special case of a correspondence that is single-valued for each  $x$ .



# Continuity for Correspondences

We want to talk about continuity of correspondences analogous to continuity of functions. What should continuity mean?

We will discuss three main notions of continuity for correspondences, each of which can be motivated by thinking about what continuity means for a function  $f : \mathbf{R}^n \rightarrow \mathbf{R}$ .

## Continuity for Correspondences

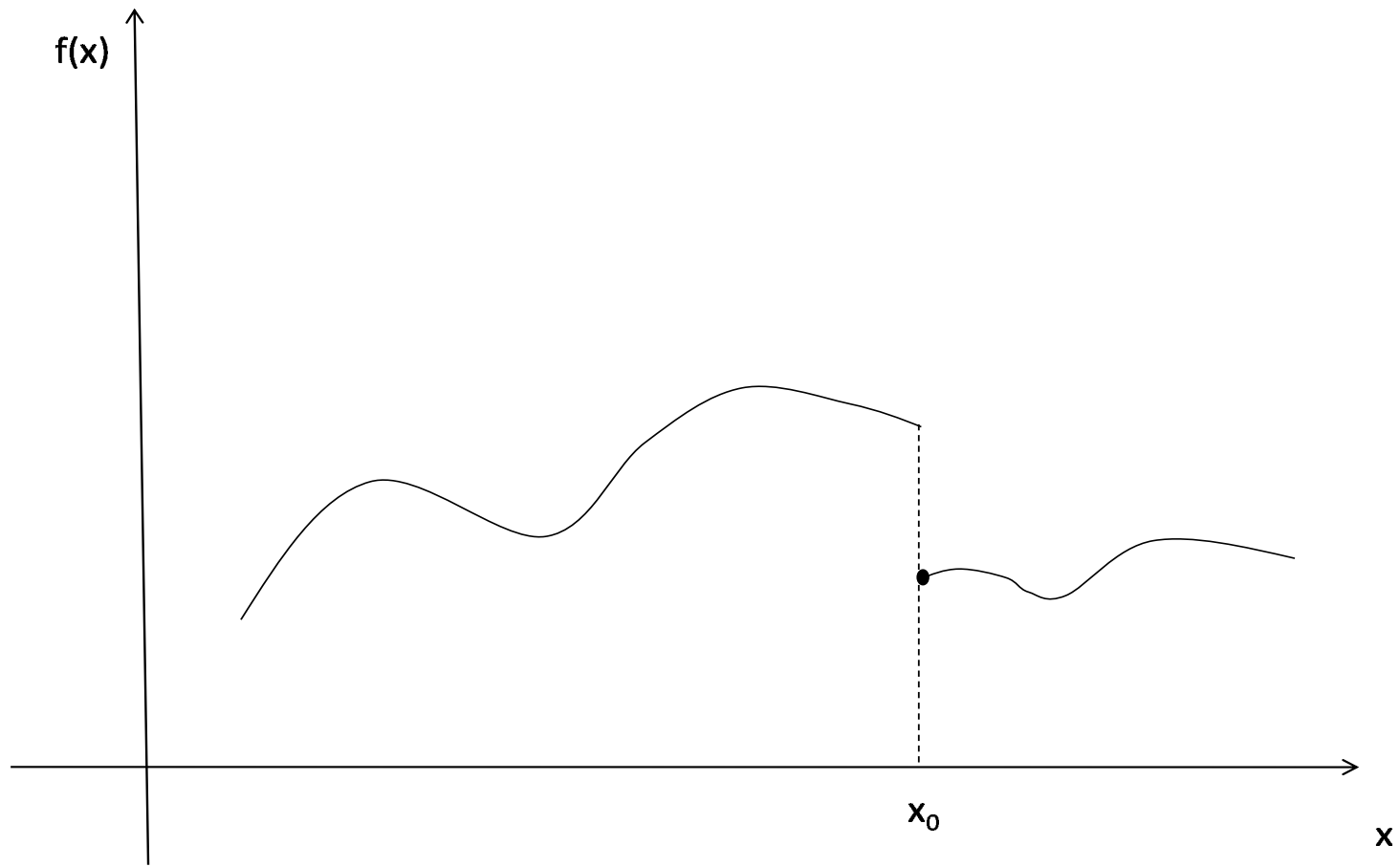
One way a function  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  may be discontinuous at a point  $x_0$  is that it “jumps downward at the limit:”

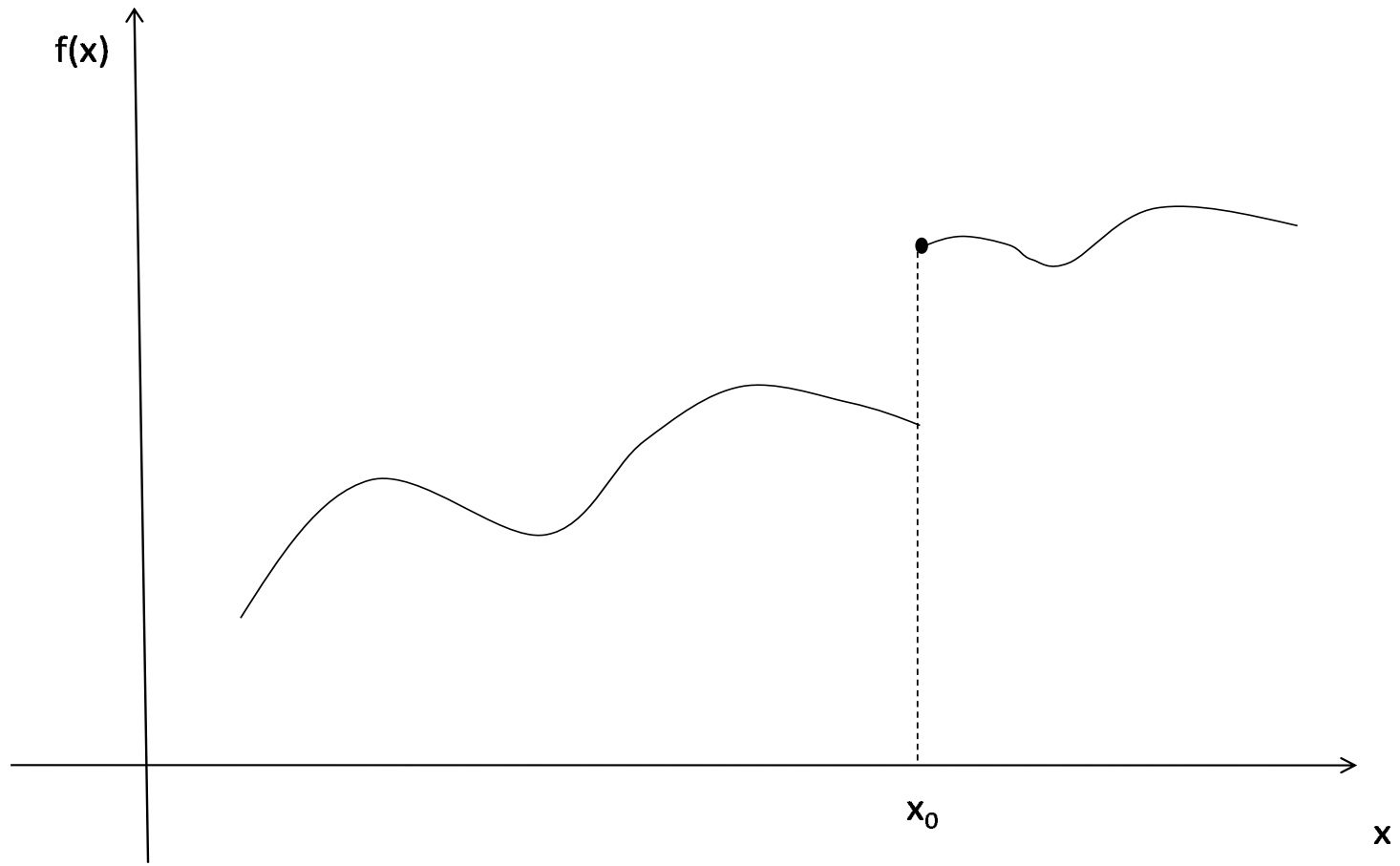
$$\exists x_n \rightarrow x_0 \text{ s.t. } f(x_0) < \liminf f(x_n)$$

It could also “jump upward at the limit:”

$$\exists x_n \rightarrow x_0 \text{ s.t. } f(x_0) > \limsup f(x_n)$$

In either case, it doesn't matter whether the sequence  $x_n$  approaches  $x_0$  from the left or the right (or both).

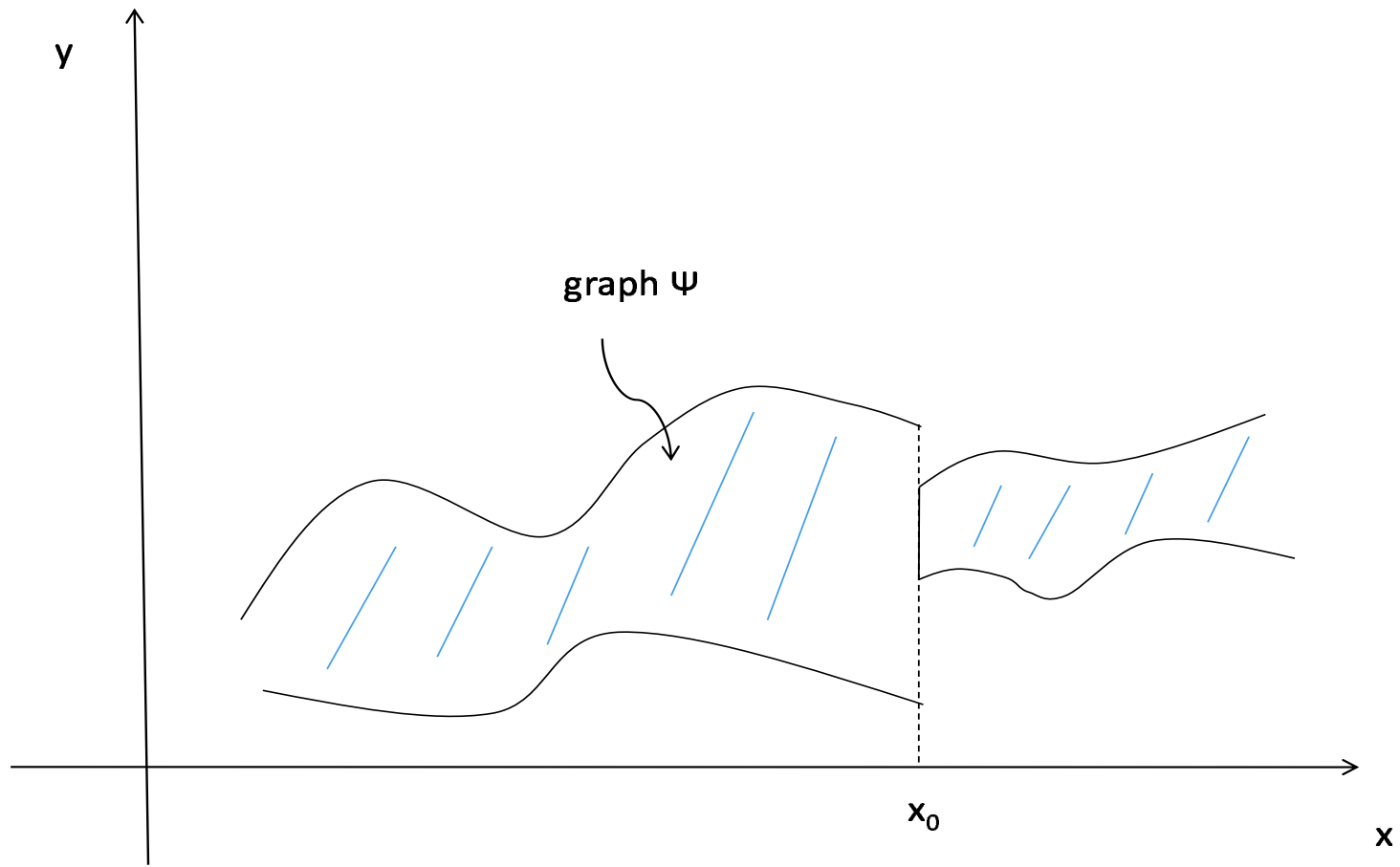




## Continuity for Correspondences

What should it mean for a *set* to “jump down” at the limit  $x_0$ ?

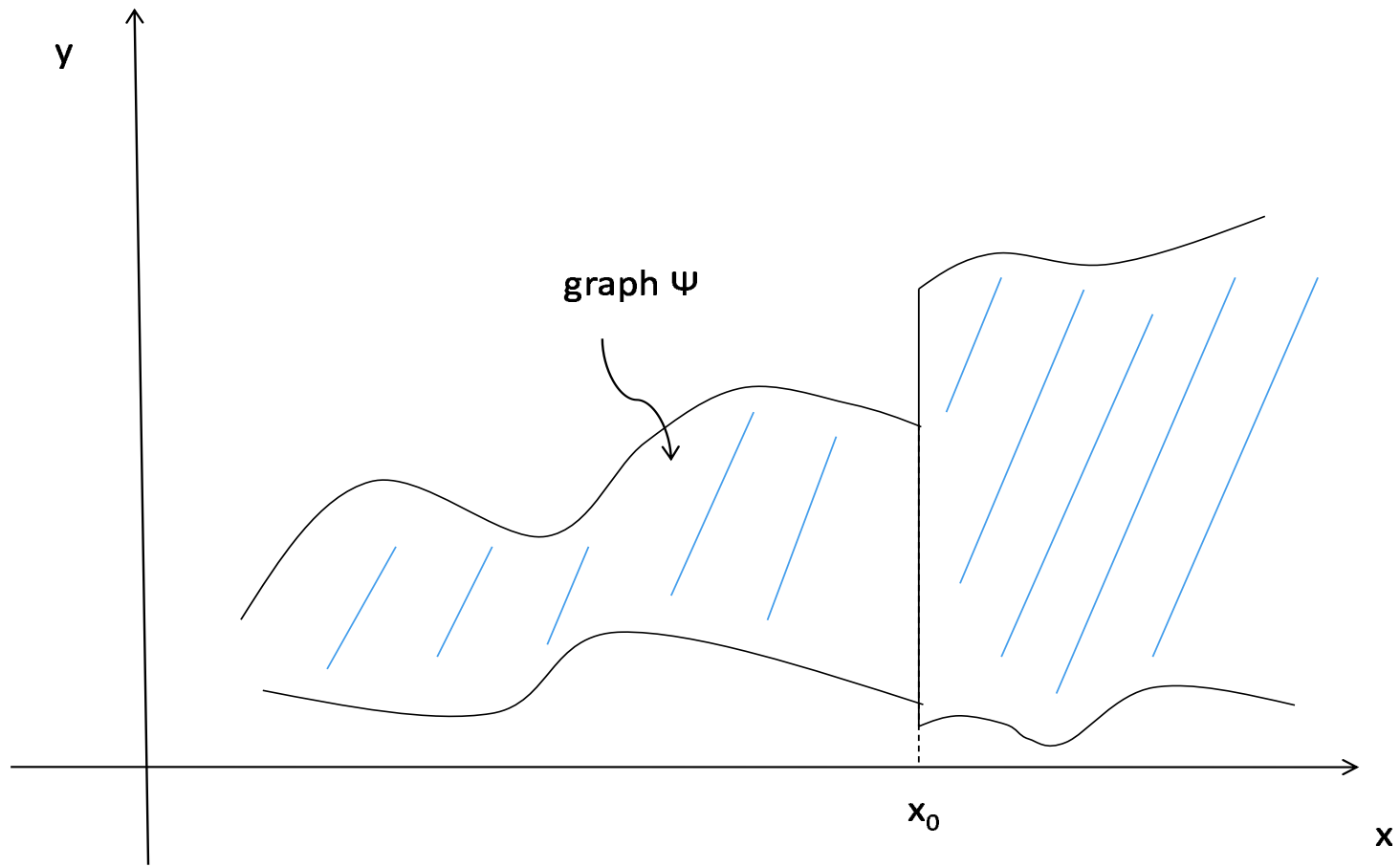
It should mean the set suddenly gets smaller – it “implodes in the limit” – that is, there is a sequence  $x_n \rightarrow x_0$  and points  $y_n \in \Psi(x_n)$  that are far from every point of  $\Psi(x_0)$  as  $n \rightarrow \infty$ .



## Continuity for Correspondences

Similarly, what should it mean for a set to “jump up” at the limit?

This should mean that the set suddenly gets bigger – it “explodes in the limit” – that is, there is a point  $y$  in  $\Psi(x_0)$  and a sequence  $x_n \rightarrow x_0$  such that  $y$  is far from every point of  $\Psi(x_n)$  as  $n \rightarrow \infty$ .





## Continuity for Correspondences

**Definition 3.** Let  $X \subseteq \mathbf{E}^n$ ,  $Y \subseteq \mathbf{E}^m$ , and  $\Psi : X \rightarrow 2^Y$ .

- $\Psi$  is upper hemicontinuous (uhc) at  $x_0 \in X$  if, for every open set  $V \supseteq \Psi(x_0)$ , there is an open set  $U$  with  $x_0 \in U$  such that

$$\Psi(x) \subseteq V \text{ for every } x \in U \cap X$$

- $\Psi$  is lower hemicontinuous (lhc) at  $x_0 \in X$  if, for every open set  $V$  such that  $\Psi(x_0) \cap V \neq \emptyset$ , there is an open set  $U$  with  $x_0 \in U$  such that

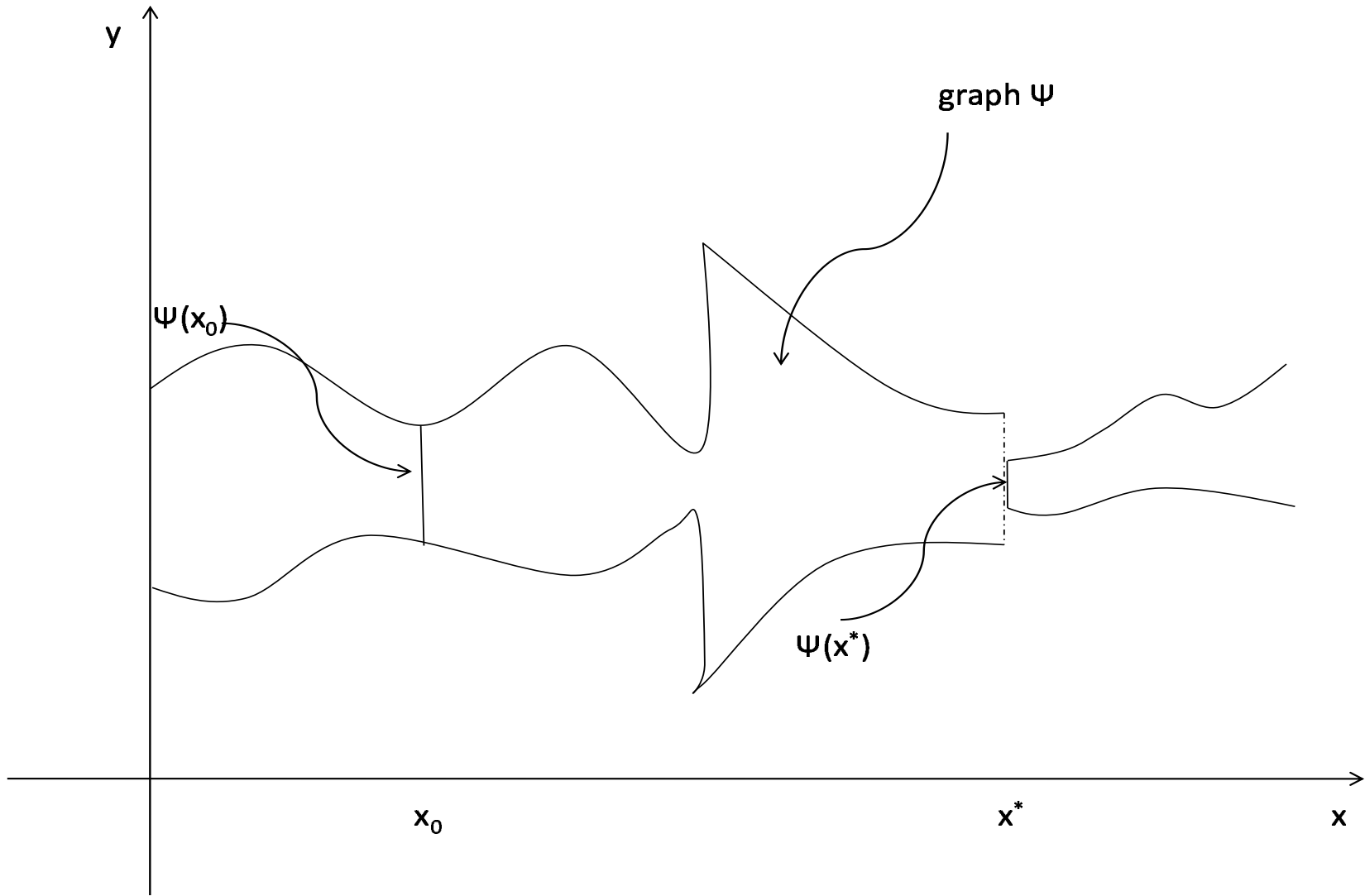
$$\Psi(x) \cap V \neq \emptyset \text{ for every } x \in U \cap X$$

- $\psi$  is continuous at  $x_0 \in X$  if it is both *uhc* and *lhc* at  $x_0$ .
- $\psi$  is upper hemicontinuous (respectively lower hemicontinuous, continuous) if it is *uhc* (respectively *lhc*, continuous) at every  $x \in X$ .

## Continuity for Correspondences

Upper hemicontinuity reflects the requirement that  $\Psi$  doesn't "implode in the limit" at  $x_0$ ; lower hemicontinuity reflects the requirement that  $\Psi$  doesn't "explode in the limit" at  $x_0$ .

Notice that upper and lower hemicontinuity are not nested: a correspondence can be upper hemicontinuous but not lower hemicontinuous, or lower hemicontinuous but not upper hemicontinuous.



## Continuity for Correspondences

An alternative notion of continuity looks instead at properties of the graph of the correspondence.

The *graph* of a correspondence  $\Psi : X \rightarrow 2^Y$  is the set

$$\text{graph } \Psi = \{(x, y) \in X \times Y : y \in \Psi(x)\}$$

## Continuity for Correspondences

Recall that a function  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  is continuous if and only if whenever  $x_n \rightarrow x$ ,  $f(x_n) \rightarrow f(x)$ . We can translate this into a statement about its graph.

Suppose  $\{(x_n, y_n)\} \subseteq \text{graph } f$  and  $(x_n, y_n) \rightarrow (x, y)$ . Since  $f$  is a function,  $(x_n, y_n) \in \text{graph } f \iff y_n = f(x_n)$ .

$$\begin{aligned} \text{So } f \text{ is continuous} &\implies y = \lim y_n = \lim f(x_n) = f(x) \\ &\implies (x, y) \in \text{graph } f \end{aligned}$$

So if  $f$  is continuous then each convergent sequence  $\{(x_n, y_n)\}$  in  $\text{graph } f$  converges to a point  $(x, y)$  in  $\text{graph } f$ , that is,  $\text{graph } f$  is closed.

## Closed Graph

**Definition 4.** Let  $X \subseteq \mathbf{E}^n$ ,  $Y \subseteq \mathbf{E}^m$ . A correspondence  $\Psi : X \rightarrow 2^Y$  has closed graph if its graph is a closed subset of  $X \times Y$ , that is, if for any sequences  $\{x_n\} \subseteq X$  and  $\{y_n\} \subseteq Y$  such that  $x_n \rightarrow x \in X$ ,  $y_n \rightarrow y \in Y$  and  $y_n \in \Psi(x_n)$  for each  $n$ , then  $y \in \Psi(x)$ .

## Closed Graph

**Example:** Consider the correspondence

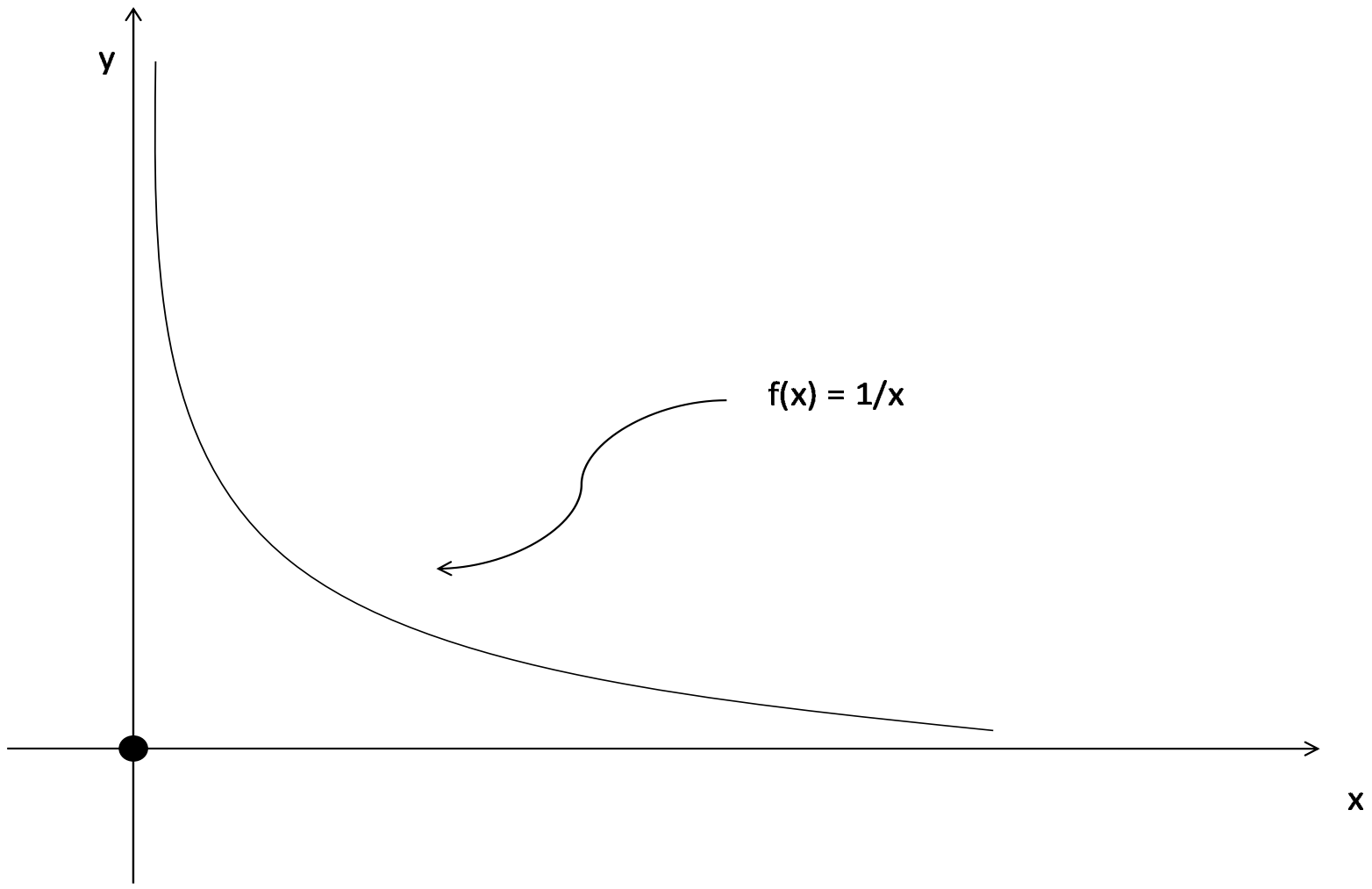
$$\Psi(x) = \begin{cases} \left\{ \frac{1}{x} \right\} & \text{if } x \in (0, 1] \\ \{0\} & \text{if } x = 0 \end{cases}$$

Let  $V = (-0.1, 0.1)$ . Then  $\Psi(0) = \{0\} \subset V$ , but no matter how close  $x$  is to 0,

$$\Psi(x) = \left\{ \frac{1}{x} \right\} \not\subset V$$

so  $\Psi$  is not uhc at 0. However, note that  $\Psi$  has closed graph.



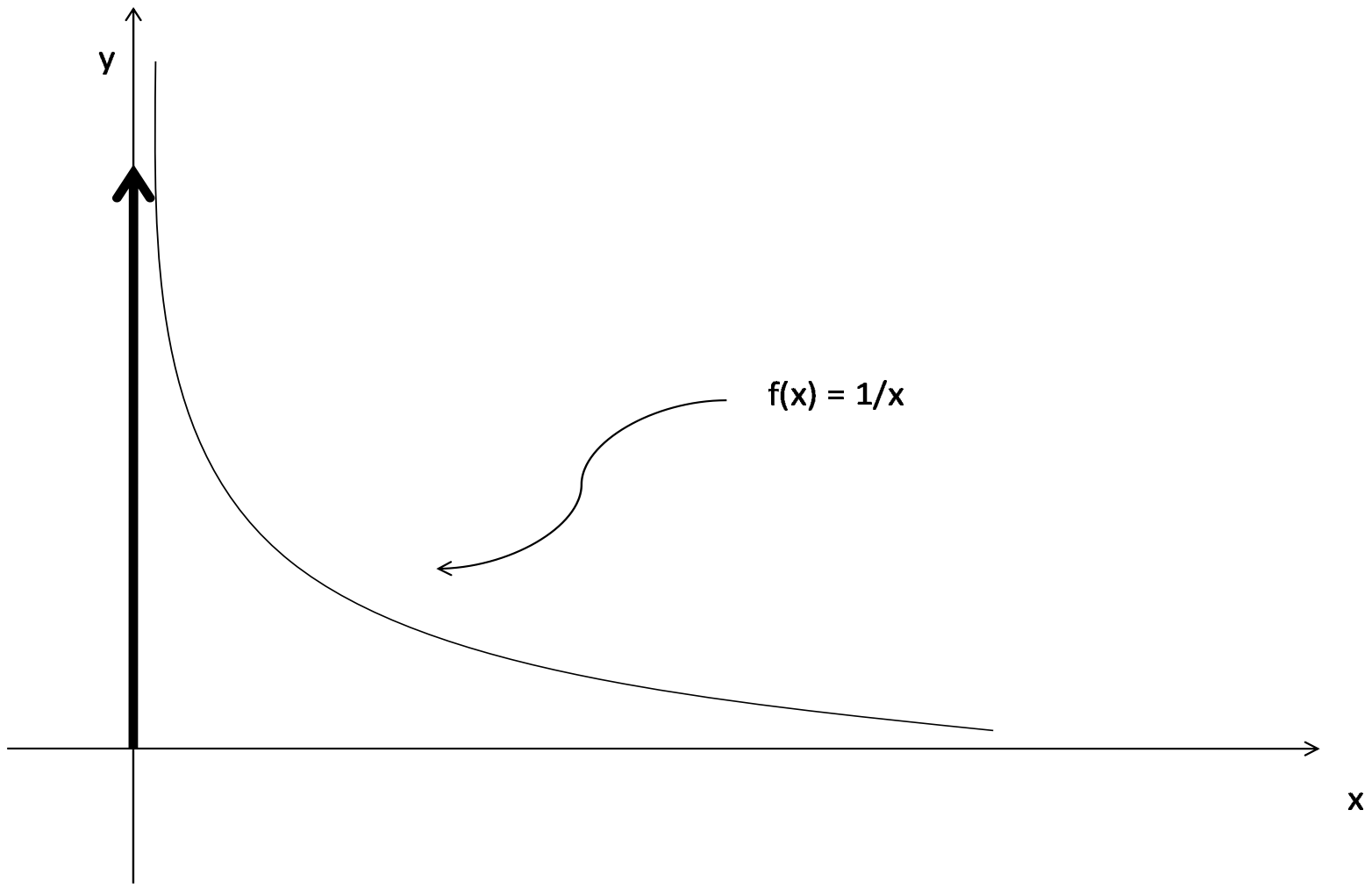


## Continuity for Correspondences

**Example:** Consider the correspondence

$$\Psi(x) = \begin{cases} \left\{ \frac{1}{x} \right\} & \text{if } x \in (0, 1] \\ \mathbf{R}_+ & \text{if } x = 0 \end{cases}$$

$\Psi(0) = [0, \infty)$ , and  $\Psi(x) \subseteq \Psi(0)$  for every  $x \in [0, 1]$ . So if  $V \supseteq \Psi(0)$  then  $V \supseteq \Psi(x)$  for all  $x$ . Thus,  $\Psi$  is uhc, and has closed graph.



# Upper Hemicontinuity and Closed Graph

For a function, upper hemi-continuity and continuity coincide.

**Theorem 3.** *Let  $X \subseteq \mathbf{E}^n$ ,  $Y \subseteq \mathbf{E}^m$  and  $f : X \rightarrow Y$ . Let  $\Psi : X \rightarrow 2^Y$  be the correspondence given by  $\Psi(x) = \{f(x)\}$  for all  $x \in X$ . Then  $\Psi$  is uhc if and only if  $f$  is continuous.*

*Proof.* Suppose  $\Psi$  is uhc. We consider the metric spaces  $(X, d)$  and  $(Y, d)$ , where  $d$  is the Euclidean metric. Fix  $V$  open in  $Y$ . Then

$$\begin{aligned} f^{-1}(V) &= \{x \in X : f(x) \in V\} \\ &= \{x \in X : \Psi(x) \subseteq V\} \end{aligned}$$

Thus,  $f$  is continuous if and only if  $f^{-1}(V)$  is open in  $X$  for each open  $V$  in  $Y$ , if and only if  $\{x \in X : \psi(x) \subseteq V\}$  is open in  $X$  for each open  $V$  in  $Y$ , if and only if  $\psi$  is uhc (as an exercise, think through why this last equivalence holds).  $\square$

# Continuity for Correspondences

For a general correspondence, these notions are not nested:

- A closed graph correspondence need not be uhc, as the first example above illustrates.
- Conversely an uhc correspondence need not have closed graph, or even have closed values.

## Continuity for Correspondences

**Definition 5.** A correspondence  $\Psi : X \rightarrow 2^Y$  is called closed-valued if  $\Psi(x)$  is a closed subset of  $Y$  for all  $x$ ;  $\Psi$  is called compact-valued if  $\Psi(x)$  is compact for all  $x$ .

For closed-valued correspondences these concepts can be more tightly connected.

- A closed-valued and upper hemicontinuous correspondence must have closed graph.
- For a closed-valued correspondence with a compact range, upper hemicontinuity is equivalent to closed graph.

## Upper Hemicontinuity and Closed Graph

**Theorem 4** (Not in de la Fuente). *Suppose  $X \subseteq \mathbf{E}^n$  and  $Y \subseteq \mathbf{E}^m$ , and  $\Psi : X \rightarrow 2^Y$ .*

- (i) If  $\Psi$  is closed-valued and uhc, then  $\Psi$  has closed graph.*
- (ii) If  $\Psi$  has closed graph and there is an open set  $W$  with  $x_0 \in W$  and a compact set  $Z$  such that  $x \in W \cap X \Rightarrow \Psi(x) \subseteq Z$ , then  $\Psi$  is uhc at  $x_0$ .*
- (iii) If  $Y$  is compact, then  $\Psi$  has closed graph  $\iff \Psi$  is closed-valued and uhc.*



*Proof.* (i) Suppose  $\Psi$  is closed-valued and uhc. If  $\Psi$  does not have closed graph, we can find a sequence  $(x_n, y_n) \rightarrow (x_0, y_0)$ , where  $(x_n, y_n)$  lies in the graph of  $\Psi$  (so  $y_n \in \Psi(x_n)$ ) but  $(x_0, y_0)$  does not lie in the graph of  $\Psi$  (so  $y_0 \notin \Psi(x_0)$ ). Since  $\Psi$  is closed-valued,  $\Psi(x_0)$  is closed. Since  $y_0 \notin \Psi(x_0)$ , there is some  $\varepsilon > 0$  such that

$$\Psi(x_0) \cap B_{2\varepsilon}(y_0) = \emptyset$$

so

$$\Psi(x_0) \subseteq \mathbf{E}^m \setminus B_\varepsilon[y_0]$$

Let  $V = \mathbf{E}^m \setminus B_\varepsilon[y_0]$ . Then  $V$  is open, and  $\Psi(x_0) \subseteq V$ . Since  $\Psi$  is uhc, there is an open set  $U$  with  $x_0 \in U$  such that

$$x \in U \cap X \Rightarrow \Psi(x) \subseteq V$$

Since  $(x_n, y_n) \rightarrow (x_0, y_0)$ ,  $x_n \in U$  for  $n$  sufficiently large, so

$$y_n \in \Psi(x_n) \subseteq V$$

Thus for  $n$  sufficiently large,  $\|y_n - y_0\| \geq \varepsilon$ , which implies that  $y_n \not\rightarrow y_0$ , and  $(x_n, y_n) \not\rightarrow (x_0, y_0)$ , a contradiction. Thus  $\Psi$  is closed-graph.

(ii) Now, suppose  $\Psi$  has closed graph and there is an open set  $W$  with  $x_0 \in W$  and a compact set  $Z$  such that

$$x \in W \cap X \Rightarrow \Psi(x) \subseteq Z$$

Since  $\Psi$  has closed graph, it is closed-valued. Let  $V$  be any open set such that  $V \supseteq \Psi(x_0)$ . We need to show there exists an open set  $U$  with  $x_0 \in U$  such that

$$x \in U \cap X \Rightarrow \Psi(x) \subseteq V$$

If not, we can find a sequence  $x_n \rightarrow x_0$  and  $y_n \in \Psi(x_n)$  such that  $y_n \notin V$ . Since  $x_n \rightarrow x_0$ ,  $x_n \in W \cap X$  for all  $n$  sufficiently large, and thus  $\Psi(x_n) \subseteq Z$  for  $n$  sufficiently large. Since  $Z$  is compact, we can find a convergent subsequence  $y_{n_k} \rightarrow y'$ . Then

$$(x_{n_k}, y_{n_k}) \rightarrow (x_0, y')$$

Since  $\Psi$  has closed graph,  $y' \in \Psi(x_0)$ , so  $y' \in V$ . Since  $V$  is open,  $y_{n_k} \in V$  for all  $k$  sufficiently large, a contradiction. Thus,  $\Psi$  is uhc at  $x_0$ .

(iii) Follows from (i) and (ii).



# Sequential Characterizations

Upper and lower hemicontinuity can be given sequential characterizations that are useful in applications.

**Theorem 5** (Thm. 11.2). *Suppose  $X \subseteq \mathbf{E}^n$  and  $Y \subseteq \mathbf{E}^m$ . A compact-valued correspondence  $\Psi : X \rightarrow 2^Y$  is uhc at  $x_0 \in X$  if and only if, for every sequence  $\{x_n\} \subseteq X$  with  $x_n \rightarrow x_0$ , and every sequence  $\{y_n\}$  such that  $y_n \in \Psi(x_n)$  for every  $n$ , there is a convergent subsequence  $\{y_{n_k}\}$  such that  $\lim y_{n_k} \in \Psi(x_0)$ .*

*Proof.* See de la Fuente. □

Note that this characterization of upper hemicontinuity requires the correspondence to have compact values.

## Sequential Characterizations

**Theorem 6** (Thm. 11.3). *A correspondence  $\Psi : X \rightarrow Y$  is lhc at  $x_0 \in X$  if and only if, for every sequence  $\{x_n\} \subseteq X$  with  $x_n \rightarrow x_0$ , and every  $y_0 \in \Psi(x_0)$ , there exists a companion sequence  $\{y_n\}$  with  $y_n \in \Psi(x_n)$  for every  $n$  such that  $y_n \rightarrow y_0$ .*

*Proof.* See de la Fuente. □

