

# Econ 204 2017

## Lecture 8

### Outline

1. Bases
2. Linear Transformations
3. Isomorphisms

# Linear Combinations and Spans

**Definition 1.** Let  $X$  be a vector space over a field  $F$ . A linear combination of  $x_1, \dots, x_n \in X$  is a vector of the form

$$y = \sum_{i=1}^n \alpha_i x_i \text{ where } \alpha_1, \dots, \alpha_n \in F$$

$\alpha_i$  is the coefficient of  $x_i$  in the linear combination.

If  $V \subseteq X$ , the span of  $V$ , denoted  $\text{span } V$ , is the set of all linear combinations of elements of  $V$ .

A set  $V \subseteq X$  spans  $X$  if  $\text{span } V = X$ .

## Linear Dependence and Independence

**Definition 2.** A set  $V \subseteq X$  is linearly dependent if there exist  $v_1, \dots, v_n \in V$  and  $\alpha_1, \dots, \alpha_n \in F$  not all zero such that

$$\sum_{i=1}^n \alpha_i v_i = 0$$

A set  $V \subseteq X$  is linearly independent if it is not linearly dependent.

Thus  $V \subseteq X$  is linearly independent if and only if

$$\sum_{i=1}^n \alpha_i v_i = 0, \quad v_i \in V \quad \forall i \Rightarrow \alpha_i = 0 \quad \forall i$$

# Bases

**Definition 3.** A Hamel basis (*often just called a basis*) of a vector space  $X$  is a linearly independent set of vectors in  $X$  that spans  $X$ .

**Example:**  $\{(1, 0), (0, 1)\}$  is a basis for  $\mathbf{R}^2$  (this is the standard basis).

**Example, cont:**  $\{(1, 1), (-1, 1)\}$  is another basis for  $\mathbf{R}^2$ :

Suppose  $(x, y) = \alpha(1, 1) + \beta(-1, 1)$  for some  $\alpha, \beta \in \mathbf{R}$

$$x = \alpha - \beta$$

$$y = \alpha + \beta$$

$$x + y = 2\alpha$$

$$\Rightarrow \alpha = \frac{x + y}{2}$$

$$y - x = 2\beta$$

$$\Rightarrow \beta = \frac{y - x}{2}$$

$$(x, y) = \frac{x + y}{2}(1, 1) + \frac{y - x}{2}(-1, 1)$$

Since  $(x, y)$  is an arbitrary element of  $\mathbf{R}^2$ ,  $\{(1, 1), (-1, 1)\}$  spans  $\mathbf{R}^2$ . If  $(x, y) = (0, 0)$ ,

$$\alpha = \frac{0 + 0}{2} = 0, \quad \beta = \frac{0 - 0}{2} = 0$$

so the coefficients are all zero, so  $\{(1, 1), (-1, 1)\}$  is linearly independent. Since it is linearly independent and spans  $\mathbf{R}^2$ , it is a basis.

**Example:**  $\{(1, 0, 0), (0, 1, 0)\}$  is not a basis of  $\mathbf{R}^3$ , because it does not span  $\mathbf{R}^3$ .

**Example:**  $\{(1, 0), (0, 1), (1, 1)\}$  is not a basis for  $\mathbf{R}^2$ .

$$1(1, 0) + 1(0, 1) + (-1)(1, 1) = (0, 0)$$

so the set is not linearly independent.

# Bases

**Theorem 1** (Thm. 1.2'). *Let  $V$  be a Hamel basis for  $X$ . Then every vector  $x \in X$  has a unique representation as a linear combination of a finite number of elements of  $V$  (with all coefficients nonzero).\**

*Proof.* Let  $x \in X$ . Since  $V$  spans  $X$ , we can write

$$x = \sum_{s \in S_1} \alpha_s v_s$$

where  $S_1$  is finite,  $\alpha_s \in F$ ,  $\alpha_s \neq 0$ , and  $v_s \in V$  for each  $s \in S_1$ . Now, suppose

$$x = \sum_{s \in S_1} \alpha_s v_s = \sum_{s \in S_2} \beta_s v_s$$

\*The unique representation of 0 is  $0 = \sum_{i \in \emptyset} \alpha_i b_i$ .

where  $S_2$  is finite,  $\beta_s \in F$ ,  $\beta_s \neq 0$ , and  $v_s \in V$  for each  $s \in S_2$ .  
Let  $S = S_1 \cup S_2$ , and define

$$\begin{aligned}\alpha_s &= 0 \quad \text{for } s \in S_2 \setminus S_1 \\ \beta_s &= 0 \quad \text{for } s \in S_1 \setminus S_2\end{aligned}$$

Then

$$\begin{aligned}0 &= x - x \\ &= \sum_{s \in S_1} \alpha_s v_s - \sum_{s \in S_2} \beta_s v_s \\ &= \sum_{s \in S} \alpha_s v_s - \sum_{s \in S} \beta_s v_s \\ &= \sum_{s \in S} (\alpha_s - \beta_s) v_s\end{aligned}$$

Since  $V$  is linearly independent, we must have  $\alpha_s - \beta_s = 0$ , so  $\alpha_s = \beta_s$ , for all  $s \in S$ .

$$s \in S_1 \Leftrightarrow \alpha_s \neq 0 \Leftrightarrow \beta_s \neq 0 \Leftrightarrow s \in S_2$$



so  $S_1 = S_2$  and  $\alpha_s = \beta_s$  for  $s \in S_1 = S_2$ , so the representation is unique.  $\square$

# Bases

**Theorem 2.** *Every vector space has a Hamel basis.*

*Proof.* The proof uses the Axiom of Choice. Indeed, the theorem is equivalent to the Axiom of Choice. □

# Bases

A closely related result, from which you can derive the previous result, shows that any linearly independent set  $V$  in a vector space  $X$  can be extended to a basis of  $X$ .

**Theorem 3.** *If  $X$  is a vector space and  $V \subseteq X$  is linearly independent, then there exists a linearly independent set  $W \subseteq X$  such that*

$$V \subseteq W \subseteq \text{span } W = X$$

# Bases

**Theorem 4.** *Any two Hamel bases of a vector space  $X$  have the same cardinality (are numerically equivalent).*

*Proof.* The proof depends on the so-called Exchange Lemma, whose idea we sketch. Suppose that  $V = \{v_\lambda : \lambda \in \Lambda\}$  and  $W = \{w_\gamma : \gamma \in \Gamma\}$  are Hamel bases of  $X$ . Remove one vector  $v_{\lambda_0}$  from  $V$ , so that it no longer spans (if it did still span, then  $v_{\lambda_0}$  would be a linear combination of other elements of  $V$ , and  $V$  would not be linearly independent). If  $w_\gamma \in \text{span}(V \setminus \{v_{\lambda_0}\})$  for every  $\gamma \in \Gamma$ , then since  $W$  spans,  $V \setminus \{v_{\lambda_0}\}$  would also span, contradiction. Thus, we can choose  $\gamma_0 \in \Gamma$  such that

$$w_{\gamma_0} \notin \text{span}(V \setminus \{v_{\lambda_0}\})$$

Because  $w_{\gamma_0} \in \text{span } V$ , we can write

$$w_{\gamma_0} = \sum_{i=0}^n \alpha_i v_{\lambda_i}$$

where  $\alpha_0$ , the coefficient of  $v_{\lambda_0}$ , is not zero (if it were, then we would have  $w_{\gamma_0} \in \text{span}(V \setminus \{v_{\lambda_0}\})$ ). Since  $\alpha_0 \neq 0$ , we can solve for  $v_{\lambda_0}$  as a linear combination of  $w_{\gamma_0}$  and  $v_{\lambda_1}, \dots, v_{\lambda_n}$ , so

$$\begin{aligned} & \text{span} \left( (V \setminus \{v_{\lambda_0}\}) \cup \{w_{\gamma_0}\} \right) \\ & \supseteq \text{span } V \\ & = X \end{aligned}$$

so

$$\left( (V \setminus \{v_{\lambda_0}\}) \cup \{w_{\gamma_0}\} \right)$$

spans  $X$ . From the fact that  $w_{\gamma_0} \notin \text{span}(V \setminus \{v_{\lambda_0}\})$  one can

show that

$$\left( (V \setminus \{v_{\lambda_0}\}) \cup \{w_{\gamma_0}\} \right)$$

is linearly independent, so it is a basis of  $X$ . Repeat this process to exchange every element of  $V$  with an element of  $W$  (when  $V$  is uncountable, this is done by a process called transfinite induction). At the end, we obtain a bijection from  $V$  to  $W$ , so that  $V$  and  $W$  are numerically equivalent.  $\square$

# Dimension

**Definition 4.** *The dimension of a vector space  $X$ , denoted  $\dim X$ , is the cardinality of any basis of  $X$ .*

For  $V \subseteq X$ ,  $|V|$  denotes the cardinality of the set  $V$ .

# Dimension

**Example:** The set of all  $m \times n$  real-valued matrices is a vector space over  $\mathbf{R}$ . A basis is given by

$$\{E_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$$

where

$$(E_{ij})_{k\ell} = \begin{cases} 1 & \text{if } k = i \text{ and } \ell = j \\ 0 & \text{otherwise.} \end{cases}$$

The dimension of the vector space of  $m \times n$  matrices is  $mn$ .



# Dimension and Dependence

**Theorem 5** (Thm. 1.4). *Suppose  $\dim X = n \in \mathbf{N}$ . If  $V \subseteq X$  and  $|V| > n$ , then  $V$  is linearly dependent.*

## Dimension and Dependence

**Theorem 6** (Thm. 1.5'). *Suppose  $\dim X = n \in \mathbf{N}$ ,  $V \subseteq X$ , and  $|V| = n$ .*

- *If  $V$  is linearly independent, then  $V$  spans  $X$ , so  $V$  is a Hamel basis.*
- *If  $V$  spans  $X$ , then  $V$  is linearly independent, so  $V$  is a Hamel basis.*

# Linear Transformations

**Definition 5.** *Let  $X$  and  $Y$  be two vector spaces over the field  $F$ . We say  $T : X \rightarrow Y$  is a linear transformation if*

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2) \quad \forall x_1, x_2 \in X, \alpha_1, \alpha_2 \in F$$

Let  $L(X, Y)$  denote the set of all linear transformations from  $X$  to  $Y$ .

# Linear Transformations

**Theorem 7.**  $L(X, Y)$  is a vector space over  $F$ .

*Proof.* First, define linear combinations in  $L(X, Y)$  as follows. For  $T_1, T_2 \in L(X, Y)$  and  $\alpha, \beta \in F$ , define  $\alpha T_1 + \beta T_2$  by

$$(\alpha T_1 + \beta T_2)(x) = \alpha T_1(x) + \beta T_2(x)$$

We need to show that  $\alpha T_1 + \beta T_2 \in L(X, Y)$ .

$$\begin{aligned} & (\alpha T_1 + \beta T_2)(\gamma x_1 + \delta x_2) \\ &= \alpha T_1(\gamma x_1 + \delta x_2) + \beta T_2(\gamma x_1 + \delta x_2) \\ &= \alpha (\gamma T_1(x_1) + \delta T_1(x_2)) + \beta (\gamma T_2(x_1) + \delta T_2(x_2)) \\ &= \gamma (\alpha T_1(x_1) + \beta T_2(x_1)) + \delta (\alpha T_1(x_2) + \beta T_2(x_2)) \\ &= \gamma (\alpha T_1 + \beta T_2)(x_1) + \delta (\alpha T_1 + \beta T_2)(x_2) \end{aligned}$$

so  $\alpha T_1 + \beta T_2 \in L(X, Y)$ .

The rest of the proof involves straightforward checking of the vector space axioms. □

# Compositions of Linear Transformations

Given  $R \in L(X, Y)$  and  $S \in L(Y, Z)$ ,  $S \circ R : X \rightarrow Z$ . We will show that  $S \circ R \in L(X, Z)$ , that is, the composition of two linear transformations is linear.

$$\begin{aligned}(S \circ R)(\alpha x_1 + \beta x_2) &= S(R(\alpha x_1 + \beta x_2)) \\ &= S(\alpha R(x_1) + \beta R(x_2)) \\ &= \alpha S(R(x_1)) + \beta S(R(x_2)) \\ &= \alpha(S \circ R)(x_1) + \beta(S \circ R)(x_2)\end{aligned}$$

so  $S \circ R \in L(X, Z)$ .

## Kernel and Rank

**Definition 6.** Let  $T \in L(X, Y)$ .

- The image of  $T$  is  $\text{Im } T = T(X)$
- The kernel of  $T$  is  $\ker T = \{x \in X : T(x) = 0\}$
- The rank of  $T$  is  $\text{Rank } T = \dim(\text{Im } T)$

# Rank-Nullity Theorem

**Theorem 8** (Thms. 2.9, 2.7, 2.6: The Rank-Nullity Theorem).  
*Let  $X$  be a finite-dimensional vector space,  $T \in L(X, Y)$ . Then  $\text{Im } T$  and  $\ker T$  are vector subspaces of  $Y$  and  $X$  respectively, and*

$$\dim X = \dim \ker T + \text{Rank } T$$



## Kernel and Rank

**Theorem 9** (Thm. 2.13).  $T \in L(X, Y)$  is one-to-one if and only if  $\ker T = \{0\}$ .

*Proof.* Suppose  $T$  is one-to-one. Suppose  $x \in \ker T$ . Then  $T(x) = 0$ . But since  $T$  is linear,  $T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0$ . Since  $T$  is one-to-one,  $x = 0$ , so  $\ker T = \{0\}$ .

Conversely, suppose that  $\ker T = \{0\}$ . Suppose  $T(x_1) = T(x_2)$ . Then

$$\begin{aligned} T(x_1 - x_2) &= T(x_1) - T(x_2) \\ &= 0 \end{aligned}$$

which says  $x_1 - x_2 \in \ker T$ , so  $x_1 - x_2 = 0$ , so  $x_1 = x_2$ . Thus,  $T$  is one-to-one.  $\square$

## Invertible Linear Transformations

**Definition 7.**  $T \in L(X, Y)$  is invertible if there exists a function  $S : Y \rightarrow X$  such that

$$S(T(x)) = x \quad \forall x \in X$$

$$T(S(y)) = y \quad \forall y \in Y$$

Denote  $S$  by  $T^{-1}$ .

Note that  $T$  is invertible if and only if it is one-to-one and onto. This is just the condition for the existence of an inverse *function*. The linearity of the inverse follows from the linearity of  $T$ .

# Invertible Linear Transformations

**Theorem 10** (Thm. 2.11). *If  $T \in L(X, Y)$  is invertible, then  $T^{-1} \in L(Y, X)$ , i.e.  $T^{-1}$  is linear.*

*Proof.* Suppose  $\alpha, \beta \in F$  and  $v, w \in Y$ . Since  $T$  is invertible, there exist unique  $v', w' \in X$  such that

$$\begin{aligned} T(v') &= v & T^{-1}(v) &= v' \\ T(w') &= w & T^{-1}(w) &= w' \end{aligned} \cdot$$

Then

$$\begin{aligned} T^{-1}(\alpha v + \beta w) &= T^{-1}(\alpha T(v') + \beta T(w')) \\ &= T^{-1}(T(\alpha v' + \beta w')) \\ &= \alpha v' + \beta w' \\ &= \alpha T^{-1}(v) + \beta T^{-1}(w) \end{aligned}$$

so  $T^{-1} \in L(Y, X)$ .



# Linear Transformations and Bases

**Theorem 11** (Thm. 3.2). *Let  $X$  and  $Y$  be two vector spaces over the same field  $F$ , and let  $V = \{v_\lambda : \lambda \in \Lambda\}$  be a basis for  $X$ . Then a linear transformation  $T \in L(X, Y)$  is completely determined by its values on  $V$ , that is:*

1. *Given any set  $\{y_\lambda : \lambda \in \Lambda\} \subseteq Y$ ,  $\exists T \in L(X, Y)$  s.t.*

$$T(v_\lambda) = y_\lambda \quad \forall \lambda \in \Lambda$$

2. *If  $S, T \in L(X, Y)$  and  $S(v_\lambda) = T(v_\lambda)$  for all  $\lambda \in \Lambda$ , then  $S = T$ .*

*Proof.* 1. If  $x \in X$ ,  $x$  has a unique representation of the form

$$x = \sum_{i=1}^n \alpha_i v_{\lambda_i} \quad \alpha_i \neq 0 \quad i = 1, \dots, n$$

(Recall that if  $x = 0$ , then  $n = 0$ .) Define

$$T(x) = \sum_{i=1}^n \alpha_i y_{\lambda_i}$$

Then  $T(x) \in Y$ . The verification that  $T$  is linear is left as an exercise.

2. Suppose  $S(v_\lambda) = T(v_\lambda)$  for all  $\lambda \in \Lambda$ . Given  $x \in X$ ,

$$\begin{aligned} S(x) &= S\left(\sum_{i=1}^n \alpha_i v_{\lambda_i}\right) \\ &= \sum_{i=1}^n \alpha_i S(v_{\lambda_i}) \\ &= \sum_{i=1}^n \alpha_i T(v_{\lambda_i}) \\ &= T\left(\sum_{i=1}^n \alpha_i v_{\lambda_i}\right) \\ &= T(x) \end{aligned}$$

so  $S = T$ .



# Isomorphisms

**Definition 8.** *Two vector spaces  $X$  and  $Y$  over a field  $F$  are isomorphic if there is an invertible  $T \in L(X, Y)$ .*

*$T \in L(X, Y)$  is an isomorphism if it is invertible (one-to-one and onto).*

Isomorphic vector spaces are essentially indistinguishable as vector spaces.



# Isomorphisms

**Theorem 12** (Thm. 3.3). *Two vector spaces  $X$  and  $Y$  over the same field are isomorphic if and only if  $\dim X = \dim Y$ .*

*Proof.* Suppose  $X$  and  $Y$  are isomorphic, and let  $T \in L(X, Y)$  be an isomorphism. Let

$$U = \{u_\lambda : \lambda \in \Lambda\}$$

be a basis of  $X$ , and let  $v_\lambda = T(u_\lambda)$  for each  $\lambda \in \Lambda$ . Set

$$V = \{v_\lambda : \lambda \in \Lambda\}$$

Since  $T$  is one-to-one,  $U$  and  $V$  have the same cardinality. If

$y \in Y$ , then there exists  $x \in X$  such that

$$\begin{aligned} y &= T(x) \\ &= T\left(\sum_{i=1}^n \alpha_{\lambda_i} u_{\lambda_i}\right) \\ &= \sum_{i=1}^n \alpha_{\lambda_i} T(u_{\lambda_i}) \\ &= \sum_{i=1}^n \alpha_{\lambda_i} v_{\lambda_i} \end{aligned}$$

which shows that  $V$  spans  $Y$ . To see that  $V$  is linearly indepen-

dent, suppose

$$\begin{aligned} 0 &= \sum_{i=1}^m \beta_i v_{\lambda_i} \\ &= \sum_{i=1}^m \beta_i T(u_{\lambda_i}) \\ &= T\left(\sum_{i=1}^m \beta_i u_{\lambda_i}\right) \end{aligned}$$

Since  $T$  is one-to-one,  $\ker T = \{0\}$ , so

$$\sum_{i=1}^m \beta_i u_{\lambda_i} = 0$$

Since  $U$  is a basis, we have  $\beta_1 = \cdots = \beta_m = 0$ , so  $V$  is linearly independent. Thus,  $V$  is a basis of  $Y$ ; since  $U$  and  $V$  are numerically equivalent,  $\dim X = \dim Y$ .

Now suppose  $\dim X = \dim Y$ . Let

$$U = \{u_\lambda : \lambda \in \Lambda\} \text{ and } V = \{v_\lambda : \lambda \in \Lambda\}$$

be bases of  $X$  and  $Y$ ; note we can use the same index set  $\Lambda$  for both because  $\dim X = \dim Y$ . By Theorem 3.2, there is a unique

$T \in L(X, Y)$  such that  $T(u_\lambda) = v_\lambda$  for all  $\lambda \in \Lambda$ . If  $T(x) = 0$ , then

$$\begin{aligned} 0 &= T(x) \\ &= T\left(\sum_{i=1}^n \alpha_i u_{\lambda_i}\right) \\ &= \sum_{i=1}^n \alpha_i T(u_{\lambda_i}) \\ &= \sum_{i=1}^n \alpha_i v_{\lambda_i} \\ &\Rightarrow \alpha_1 = \cdots = \alpha_n = 0 \text{ since } V \text{ is a basis} \\ &\Rightarrow x = 0 \\ &\Rightarrow \ker T = \{0\} \\ &\Rightarrow T \text{ is one-to-one} \end{aligned}$$

If  $y \in Y$ , write  $y = \sum_{i=1}^m \beta_i v_{\lambda_i}$ . Let

$$x = \sum_{i=1}^m \beta_i u_{\lambda_i}$$

Then

$$\begin{aligned} T(x) &= T\left(\sum_{i=1}^m \beta_i u_{\lambda_i}\right) \\ &= \sum_{i=1}^m \beta_i T(u_{\lambda_i}) \\ &= \sum_{i=1}^m \beta_i v_{\lambda_i} \\ &= y \end{aligned}$$

so  $T$  is onto, so  $T$  is an isomorphism and  $X, Y$  are isomorphic.  $\square$