Econ 204 – Problem Set 1

Due Friday July 21, 2017

1. Use induction to prove the following:
   (a) $2^{2n} - 1$ is divisible by 3 for all $n \in \mathbb{N}$.
   (b) $1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n}$

2. In the following examples, show that the sets A and B are numerically equivalent by finding a specific bijection between the two.
   (a) $A = [0,1],\ B = [10,20]$
   (b) $A = [0,1],\ B = [0,1)$
   (c) $A = (-1,1),\ B = \mathbb{R}$

3. Define the infinite **cartesian product** of a set X with itself as $X^\omega := \prod_{i\in\mathbb{N}} X$. Prove by contradiction that for $X = \{0,1\}$, X^ω is uncountable. (Hint: suppose there exists a surjective map $f : \mathbb{N} \to X^\omega$, and find an element in X^ω which is not in the image of f).

4. **(Dynkins's π-λ system Theorem)**: The goal of this exercise is to prove this theorem, and review/practice some of the set theoretical results.

   Suppose Ω is some arbitrary set (which need not have any topological or algebraic structure). Then, $\mathscr{F} \subset 2^\Omega$ as a collection of subsets of Ω is called a **σ-algebra** if it satisfies following properties:

   - $\emptyset \in \mathscr{F}$.
   - For every $A \subset \Omega$ where $A \in \mathscr{F}$, $A^c \in \mathscr{F}$ (A^c refers to the complement of set A, i.e $\Omega \setminus A$).
   - For every *countable* sequence of subsets $\{A_n\}$ where $A_n \in \mathscr{F}$ for all n, $\bigcup_n A_n \in \mathscr{F}$.

   (a) Show that $\Omega \in \mathscr{F}$.
   (b) Prove that $\mathscr{F}$ is closed under countable intersection.

   Two more definitions: first, $\Lambda \subset 2^\Omega$ is called a **λ-system** if:

   - $\Omega \in \Lambda$.
   - If $A, B \in \Lambda$ and $A \subset B$, then $B \setminus A \in \Lambda$.
   - If $\{A_n\}$ is an increasing sequence of subsets, i.e $A_1 \subset A_2 \subset \ldots$, with each element being in Λ, then $\bigcup_n A_n \in \Lambda$.

   Second, $\Pi \subset 2^\Omega$ is called a **π-system**, if it is closed under *finite* intersection. Now **assume** Π is a π-system such that $\Pi \subset \Lambda$, where Λ is a λ-system. The Dynkin's theorem which we want to prove states that the smallest σ-algebra containing Π (denoted by $\sigma(\Pi)$) is a subset of Λ. Try to keep on with each step below until the final result drops out:

(c) Let $\lambda(\Pi)$ be the *smallest*[1] λ-system containing Π. Explain why $\lambda(\Pi) \subset \Lambda$. (Hint: note that $\Pi \subset \Lambda$).

(d) Let $B \in \Pi$ and define $\mathcal{A}_B := \{A \subset \Omega : A \cap B \in \lambda(\Pi)\}$. Show that $\mathcal{A}_B$ is itself a λ-system and contains $\lambda(\Pi)$, i.e $\lambda(\Pi) \subset \mathcal{A}_B$.

(e) Now let $A \in \lambda(\Pi)$, and define $\mathcal{B}_A := \{B \subset \Omega : A \cap B \in \lambda(\Pi)\}$. Again, show that $\mathcal{B}_A$ is a λ-system containing $\lambda(\Pi)$.

(f) Use the previous two steps to show $\lambda(\Pi)$ is also a π-system.

(g) Given that we have seen so far that $\lambda(\Pi)$ is both a π-system and a λ-system, deduce that it has to be a σ-algebra.

(h) Conclude that $\sigma(\Pi) \subset \Lambda$.

5. Let S be some arbitrary space, and $f, g : S \to \mathbb{R}$ be real-valued functions defined on S, such that $|f(s)| \leq M_1$ and $|g(s)| \leq M_2$ for all $s \in S$. Answer the following items:

   (a) For the case where $S = \left\{\frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$ and $f(x) = e^x$ show that $f(S)$ is bounded from above and below. Then, find its supremum and infimum. Are they equal to the maximum and minimum of f on S?

   (b) Back to the general setup, show that:

$$\sup_{s\in S}\{-f(s)\} = -\inf_{s\in S}\{f(s)\} \tag{1}$$

   (c) Again for the general case, prove that:

$$\sup_{s\in S}\big(f(s) + g(s)\big) \leq \sup_{s\in S} f(s) + \sup_{s\in S} g(s) \tag{2}$$

   (d) Now suppose $S = U \times V$, i.e cartesian product of U and V. Prove the following inequality:

$$\inf_{u\in U}\sup_{v\in V} f(u,v) \geq \sup_{v\in V}\inf_{u\in U} f(u,v) \tag{3}$$

6. Suppose a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ has the following properties for all x, y:

   - $f(x) = 0 \iff x = 0$
   - $x \geq y \implies f(x) \geq f(y)$
   - $f(x+y) \leq f(x) + f(y)$

   Show that if (X, d) is a metric space, then $(X, f \circ d)$ is also a metric space.

[1] By smallest we mean: $\lambda(\Pi) = \bigcap_{\{\Lambda_\alpha \text{ is } \lambda\text{-system in } 2^\Omega : \Pi \subset \Lambda_\alpha\}} \Lambda_\alpha$