1. Use induction to prove the following:
   (a) \(2^{2n} - 1\) is divisible by 3 for all \(n \in \mathbb{N}\).
   (b) \(1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n}\)

2. In the following examples, show that the sets \(A\) and \(B\) are numerically equivalent by finding a specific bijection between the two.
   (a) \(A = [0, 1], B = [10, 20]\)
   (b) \(A = [0, 1], B = [0, 1)\)
   (c) \(A = (-1, 1), B = \mathbb{R}\)

3. Define the infinite cartesian product of a set \(X\) with itself as \(X^\omega := \prod_{i \in \mathbb{N}} X\). Prove by contradiction that for \(X = \{0, 1\}\), \(X^\omega\) is uncountable. (Hint: suppose there exists a surjective map \(f : \mathbb{N} \rightarrow X^\omega\), and find an element in \(X^\omega\) which is not in the image of \(f\)).

4. (Dynkin’s \(\pi\)-\(\lambda\) system Theorem): The goal of this exercise is to prove this theorem, and review/practice some of the set theoretical results.
   Suppose \(\Omega\) is some arbitrary set (which need not have any topological or algebraic structure). Then, \(\mathcal{F} \subset 2^\Omega\) as a collection of subsets of \(\Omega\) is called a \(\sigma\)-algebra if it satisfies following properties:
   - \(\emptyset \in \mathcal{F}\).
   - For every \(A \subset \Omega\) where \(A \in \mathcal{F}\), \(A^c \in \mathcal{F}\) (\(A^c\) refers to the complement of set \(A\), i.e \(\Omega \setminus A\)).
   - For every countable sequence of subsets \(\{A_n\}\) where \(A_n \in \mathcal{F}\) for all \(n\), \(\bigcup_n A_n \in \mathcal{F}\).
   (a) Show that \(\Omega \in \mathcal{F}\).
   (b) Prove that \(\mathcal{F}\) is closed under countable intersection.

Two more definitions: first, \(\Lambda \subset 2^\Omega\) is called a \(\lambda\)-system if:
   - \(\Omega \in \Lambda\).
   - If \(A, B \in \Lambda\) and \(A \subset B\), then \(B \setminus A \in \Lambda\).
   - If \(\{A_n\}\) is an increasing sequence of subsets, i.e \(A_1 \subset A_2 \subset \ldots\), with each element being in \(\Lambda\), then \(\bigcup_n A_n \in \Lambda\).

Second, \(\Pi \subset 2^\Omega\) is called a \(\pi\)-system, if it is closed under finite intersection. Now assume \(\Pi\) is a \(\pi\)-system such that \(\Pi \subset \Lambda\), where \(\Lambda\) is a \(\lambda\)-system. The Dynkin’s theorem which we want to prove states that the smallest \(\sigma\)-algebra containing \(\Pi\) (denoted by \(\sigma(\Pi)\)) is a subset of \(\Lambda\). Try to keep on with each step below until the final result drops out:
(c) Let $\lambda(\Pi)$ be the smallest $\lambda$-system containing $\Pi$. Explain why $\lambda(\Pi) \subseteq \Lambda$. (Hint: note that $\Pi \subseteq \Lambda$).

(d) Let $B \in \Pi$ and define $A_B := \{ A \subseteq \Omega : A \cap B \in \lambda(\Pi) \}$. Show that $A_B$ is itself a $\lambda$-system and contains $\lambda(\Pi)$, i.e $\lambda(\Pi) \subseteq A_B$.

(e) Now let $A \in \lambda(\Pi)$, and define $B_A := \{ B \subseteq \Omega : A \cap B \in \lambda(\Pi) \}$. Again, show that $B_A$ is a $\lambda$-system containing $\lambda(\Pi)$.

(f) Use the previous two steps to show $\lambda(\Pi)$ is also a $\pi$-system.

(g) Given that we have seen so far that $\lambda(\Pi)$ is both a $\pi$-system and a $\lambda$-system, deduce that it has to be a $\sigma$-algebra.

(h) Conclude that $\sigma(\Pi) \subseteq \Lambda$.

5. Let $S$ be some arbitrary space, and $f, g : S \to \mathbb{R}$ be real-valued functions defined on $S$, such that $|f(s)| \leq M_1$ and $|g(s)| \leq M_2$ for all $s \in S$. Answer the following items:

(a) For the case where $S = \{(\frac{(-1)^n}{n} : n \in \mathbb{N}\}$ and $f(x) = e^x$ show that $f(S)$ is bounded from above and below. Then, find its supremum and infimum. Are they equal to the maximum and minimum of $f$ on $S$?

(b) Back to the general setup, show that:

$$\sup_{s \in S} \{-f(s)\} = -\inf_{s \in S} \{f(s)\} \quad (1)$$

(c) Again for the general case, prove that:

$$\sup_{s \in S} \{f(s) + g(s)\} \leq \sup_{s \in S} f(s) + \sup_{s \in S} g(s) \quad (2)$$

(d) Now suppose $S = U \times V$, i.e cartesian product of $U$ and $V$. Prove the following inequality:

$$\inf_{u \in U} \sup_{v \in V} f(u, v) \geq \sup_{v \in V} \inf_{u \in U} f(u, v) \quad (3)$$

6. Suppose a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ has the following properties for all $x, y$:

- $f(x) = 0 \iff x = 0$
- $x \geq y \implies f(x) \geq f(y)$
- $f(x + y) \leq f(x) + f(y)$

Show that if $(X, d)$ is a metric space, then $(X, f \circ d)$ is also a metric space.

\[ ^1 \text{By smallest we mean: } \lambda(\Pi) = \bigcap \{ \Lambda_\alpha \text{ is } \lambda\text{-system in } 2^\Omega : \Pi \subseteq \Lambda_\alpha \} \Lambda_\alpha \]